Block diagrams and modeling

Problem 1. Pictured in Figure 1 is a sketch of Drebbel’s incubator. The alcohol will expand or contract depending on the temperature of the water. This, in turn, adjusts the damper.

Figure 1: Drebbel’s incubator was an early feedback control system for incubating chicken eggs. It was invented around 1620.

Draw a component block diagram for Drebbel’s incubator. Identify the system output, plant, sensors, and controller. Describe the process for each.

Solution 1. The block diagram looks just as a typical one, as you can see in Figure 2.

The output of the closed-loop system is the water temperature.

Here, the controller is the process that maps from alcohol density to how open the damper is, which includes the entire process of the pulling the mercury up and down, and the lever that affects where the damper is positioned. The control law

The sensor is the process that maps water temperature to alcohol density.

The plant is the process that takes in the damper’s position and outputs a water temperature. This process operates since the damper controls how much oxygen the fire gets, and therefore the level of heat applied to the water.

Figure 2: A general block diagram, which also applies to Drebbel’s incubator. The controller, plant, and sensor are described in the text.
Linear algebra review

**Problem 2.** Calculate the characteristic polynomial and eigenvalues for each of the matrices below.

As a reminder, the characteristic polynomial of a matrix $A$ is given by $p_A(s) = \det(sI - A)$, where $I$ is the identity matrix with dimensions matching $A$, and $sI$ is the identity matrix $I$ multiplied by a scalar $s$.

i) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

ii) $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

**Solution 2.** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $sI - A = \begin{bmatrix} s-a & -b \\ -c & s-d \end{bmatrix}$, so $\det(sI - A) = (s-a)(s-d) - (-b)(-c) = s^2 - as - ds + ad - bc = s^2 + (-a - d)s + (ad - bc)$. The eigenvalues are the roots of this polynomial. So:

i) $p_A(s) = s^2 - 2s - 24$ and $\lambda(A) = \{-4, 6\}$.

ii) $p_A(s) = s^2 - 3s + 2$ and $\lambda(A) = \{1, 2\}$.

iii) $p_A(s) = s^2 - 2s + 10$ and $\lambda(A) = \{1 - 3i, 1 + 3i\}$.

**Problem 3.** What’s the relationship between the eigenvalues of a matrix $A$ and $\det(A)$?

**Solution 3.** The determinant of $A$ equals the product of the eigenvalues of $A$. 

2
Complex numbers review

Problem 4. Calculate the magnitude and phase of the following complex numbers.

i) \( x = 3 + 4j \)

ii) \( x = 21 - 20j \)

iii) \( x = a/b \), where the magnitudes \(|a|\) and \(|b|\) are given, as well as the phases \(\angle a\) and \(\angle b\).

Solution 4. It helps to draw these out.

i) \(|x| = 5\) and \(\angle x \approx 0.92729522\) radians \(\approx 53.130102\) degrees.

ii) \(|x| = 29\) and \(\angle x \approx -0.76101275\) radians \(\approx -43.60282\) degrees \(= 316.39718\) degrees.

iii) \(|x| = |a|/|b|\) and \(\angle x = \angle a - \angle b\).
Putting ODEs in state-space form

Problem 5. Take the following ordinary differential equations (ODEs) and write them in state-space form, i.e. equations of the form $\dot{x} = Ax + Bu$.

We use the notation where $x^{(n)}$ denotes the $n$th derivative of $x(t)$. Be careful with minus signs!

\( \begin{align*}
&i) \quad x^{(5)} - x^{(4)} + 3x^{(2)} = 16x^{(1)} + 12x - 2u \\
&ii) \quad x^{(4)} + a_3x^{(3)} + a_2x^{(2)} + a_1x^{(1)} + a_0x = u. \text{ Here, each } a_i \text{ is a known constant.}
\end{align*} \)

Solution 5. \( \begin{align*}
&i) \text{ Let } x_1 = x, x_2 = x^{(1)}, x_3 = x^{(2)}, x_4 = x^{(3)}, x_5 = x^{(4)}, \text{ and note that } \dot{x}_5 = x^{(5)}. \\
&\dot{x}_1 = \quad x_2 \\
&\dot{x}_2 = \quad x_3 \\
&\dot{x}_3 = \quad x_4 \\
&\dot{x}_4 = \quad x_5 \\
&\dot{x}_5 = \quad x_5 - 3x_3 + 16x_2 + 12x_1 - 2u
\end{align*} \)

In matrix form:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
12 & 16 & -3 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
-2
\end{bmatrix}
\]

\( \begin{align*}
&ii) \text{ As before, let } x_1 = x \text{ and } x_i = x^{(i-1)}. \\
&\dot{x}_1 = \quad x_2 \\
&\dot{x}_2 = \quad x_3 \\
&\dot{x}_3 = \quad x_4 \\
&\dot{x}_4 = -a_3x_4 - a_2x_3 - a_1x_2 - a_0x_1 + u
\end{align*} \)

In matrix form:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-a_0 & -a_1 & -a_2 & -a_3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

4
Deriving dynamics from a linear circuit

**Problem 6.** Derive a state space model of the form \( \dot{x} = Ax + Bu \) that models the dynamics of the RLC circuit in Figure 3. Use \( V_S \) as the input. You may choose what your states are, but explicitly declare your choice.

[Image: A typical RLC circuit.]

**Solution 6.** Let \( v_c, v_l, v_r \) denote the voltages across the capacitor, inductor, and resistor, respectively. Also, let’s use the convention where the directionality of the voltages is such that \( v_S = v_c + v_l + v_r \), i.e. voltage drops as we go clockwise. (The previous statement is due to Kirchhoff’s voltage law (KVL).) Note that all elements have the same current: \( i = i_c = i_l = i_r \). (This is kind of trivial instantiation of Kirchhoff’s current law (KCL).)

Capacitors do not allow sudden changes in voltage; inductors do not allow sudden changes in current; we’ll use the physical quantities as our states. \( x_1 = i, x_2 = v_c \). Then, \( \dot{x}_1 = v_l/L \) and \( \dot{x}_2 = i_c/C = i/C \).

\[
\begin{align*}
\dot{x}_1 &= \frac{1}{L} v_l \\
\dot{x}_2 &= \frac{1}{C} i
\end{align*}
\]

Plugging in KVL, \( v_l = v_S - v_c - v_r \). Ohm’s law gives us that \( v_r = i_r R = iR \), so we have the relations:

\[
\begin{align*}
\dot{x}_1 &= \frac{1}{L} (v_S - v_c - iR) \\
\dot{x}_2 &= \frac{1}{C} i
\end{align*}
\]

So our input \( u = V_S \), and the other terms are states, so we finally have our state-space equation:

\[
\begin{align*}
\dot{x}_1 &= -\frac{R}{L} x_1 - \frac{1}{L} x_2 + \frac{1}{L} u \\
\dot{x}_2 &= \frac{1}{C} x_1
\end{align*}
\]

This is when \( x_1 = i, x_2 = v_c \).

Alternatively, if we define \( x_1 = v_c \) and \( x_2 = \dot{v}_c \):

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{1}{LC} x_1 - \frac{R}{L} x_2 + \frac{1}{LC} u
\end{align*}
\]
Nonlinear dynamics

Problem 7. Consider the following second-order differential equation:

\[
\frac{d^2y}{dt^2} - (1 - y^2) \frac{dy}{dt} + y = 0
\]

i) Write the dynamics as a non-linear state-space equation.

**Remark:** State-space models that do not have an input, i.e. are of the form \( \dot{x} = Ax \), are ‘autonomous’, since they evolve on their own\(^1\).

ii) Identify all equilibria of the system, i.e. points \( x \) such that \( \dot{x} = 0 \). You must both find these equilibria and argue that there are no others.

iii) For each equilibrium point, linearize your dynamics about said equilibrium point, and give the linearized dynamics in state-space form, i.e. \( \dot{x} = Ax \).

Solution 7. i) Letting \( x_1 = y \) and \( x_2 = \frac{dy}{dt} \):

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= (1 - x_1^2)x_2 - x_1
\end{align*}
\]

ii) First, we require that \( \dot{x}_1 = 0 \), which leads to the constraint \( x_2 = 0 \). This is a necessary condition for an equilibrium point.

Next, we look at \( \dot{x}_2 = 0 \), which leads to the constraint \( (1 - x_1^2)x_2 - x_1 = 0 \). Recalling the necessary constraint \( x_2 = 0 \), we can write this as \( x_1 = 0 \), which is also a necessary constraint.

Thus, we can see \( x = 0 \) is an equilibrium, and the only point that satisfies the necessary properties of an equilibrium point.

iii) We think of \( \dot{x} = f(x) \) and take the partial derivatives. For \( \dot{x}_1 = f_1(x) \), the partial derivatives are straightforward.

For \( \dot{x}_2 = f_2(x) = (1 - x_1^2)x_2 - x_1 \), we have:

\[
\begin{align*}
\frac{\partial}{\partial x_1} [(1 - x_1^2)x_2 - x_1] &= -2x_1x_2 - 1 \\
\frac{\partial}{\partial x_2} [(1 - x_1^2)x_2 - x_1] &= 1 - x_1^2
\end{align*}
\]

Evaluating these partials at \( x = 0 \) yields:

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}
\]

\(^1\)Care must be taken with this terminology, as an ‘autonomous’ system means something different to mathematicians. In mathematics, a differential equation \( \frac{dy}{dx} = f(y) \) is autonomous because the right-hand side does not depend on \( x \).