UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Department of Electrical and Computer Engineering

ECE 486: Control Systems

Homework 5 Solutions

Spring 2014

Problem 1

Sketch the root loci below by hand, applying rules A—F.

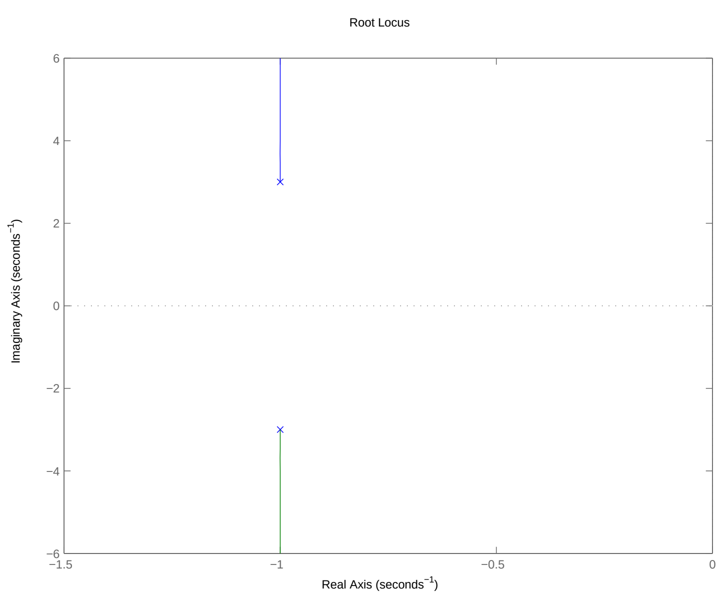
(a) $KL(s) = K\dfrac{1}{s^2+2s+10}$

OL zeros: None
OL poles: $-1 \pm 3j$

Rule A: $n = 2, m = 0 \Rightarrow$ 2 Branches.
Rule B: 2 branches start at OL poles. $n - m = 2 \Rightarrow$ 2 branches go to infinity.
Rule C: $m = 0 \Rightarrow$ No OL zeros.
Rule D: No real locus. No OL poles or zeros on the real axis.
Rule E: $n - m = 2 \Rightarrow \alpha = \dfrac{\Sigma_i^n p_i - \Sigma_j^m z_j}{n-m} = \dfrac{-1-3j-1+3j}{2} = -1$

$\phi_l = \dfrac{180^\circ + 360^\circ(l-1)}{n-m}, l = 1, .., n-m-1 = 90^\circ, 270^\circ$

Rule F: $j\omega-$crossing
RL correspond to values of K for which Routh criterion signals transition between stability and instability. However, there is no such K in this case. Therefore, there is no $j\omega-$crossing.



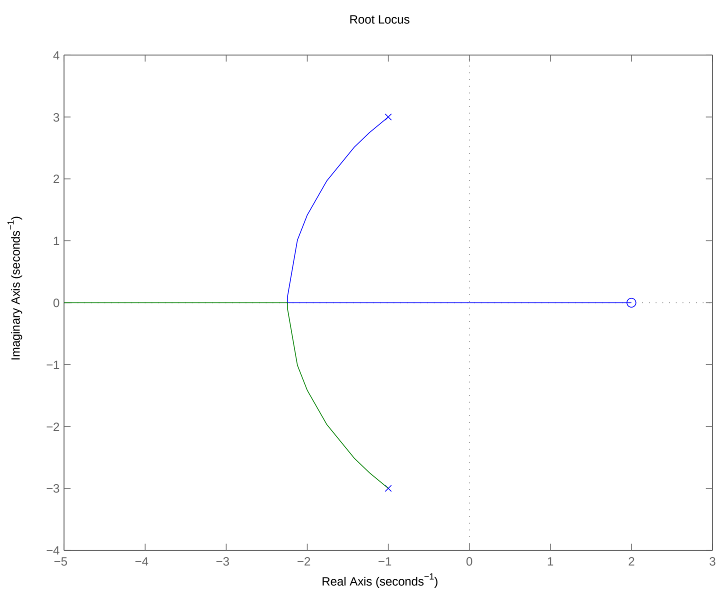
(b) $KL(s) = K\frac{s-2}{s^2+2s+10}$

OL zeros: 2
OL poles: $-1 \pm 3j$

Rule A: $n = 2, m = 1 \Rightarrow 2$ Branches.
Rule B: $n = 2 \Rightarrow 2$ branches start at OL poles. $n - m = 1 \Rightarrow 1$ branch goes to infinity.
Rule C: $m = 1 \Rightarrow 1$ branch ends at OL zeros.
Rule D: Real locus: $(-\infty, 2]$
Rule E: $n - m = 1 \Rightarrow \alpha = \frac{-1-3j-1+3j-2}{1} = -4$
$\phi_l = \frac{180^\circ + 360^\circ(l-1)}{n-m}, l = 1, .., n-m-1 = 180^\circ$
Rule F: $j\omega-$crossing

$$\begin{aligned} 0 &= s^2 + 2s + 10 + K(s-2)|_{s=j\omega} \\ &= (j\omega)^2 + (K+2)j\omega + (10-2K) \\ &= -\omega^2 + (10-2K) + (K+2)\omega j \end{aligned}$$

Solve $-\omega^2 + (10+2K) = 0$ and $(K+2)\omega = 0$
For $K > 0$, $\omega = 0$ satisfies these equalities. Therefore, there is a $j\omega$-crossing at $\omega = 0$



(c) $KL(s) = K\dfrac{(s+1)(s+2)}{s(s^2+4)(s^2+5)}$

OL zeros: $-1, -2$

OL poles: $0, \pm 2j, \pm\sqrt{5}j$

Rule A: $n = 5, m = 2 \Rightarrow 5$ Branches.

Rule B: $n = 5 \Rightarrow 5$ branches start at OL poles. $n - m = 3 \Rightarrow 3$ branches go to infinity.

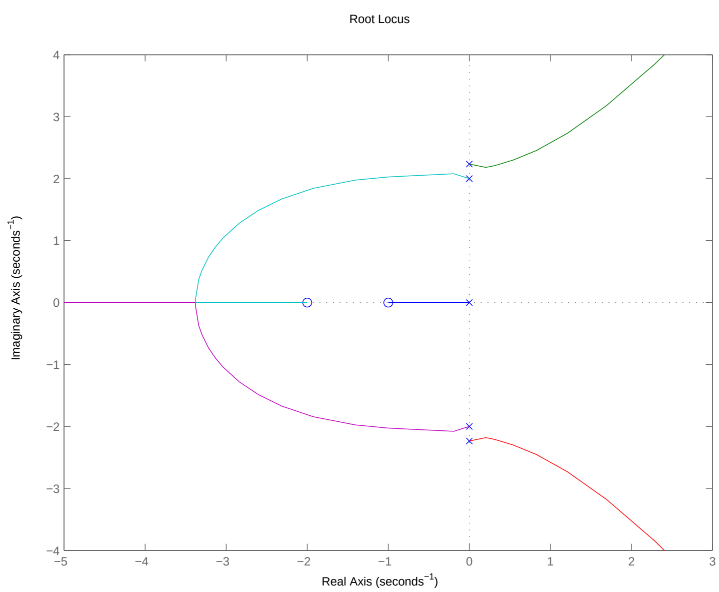
Rule C: $m = 2 \Rightarrow 2$ branches end at OL zeros.

Rule D: Real locus: $[-1, 0], (-\infty, -2]$

Rule E: $n - m = 3 \Rightarrow \alpha = \dfrac{\pm 2j \pm \sqrt{5}j - (-1-2)}{3} = 1$

$\phi_l = \dfrac{180° + 360°(l-1)}{n-m}, l = 1, .., n-m-1 = 60°, 180°, 300°$

Rule F: No $j\omega-$crossing



(d) $KL(s) = K\dfrac{s+2}{s^5+1}$

OL zeros: -2

OL poles: $-1, -0.309 \pm j0.9511, 0.809 \pm j0.5878$

Rule A: $n = 5, m = 1 \Rightarrow 5$ Branches.

Rule B: $n = 5 \Rightarrow 5$ branches start at OL poles. $n - m = 4 \Rightarrow 4$ branches go to infinity.

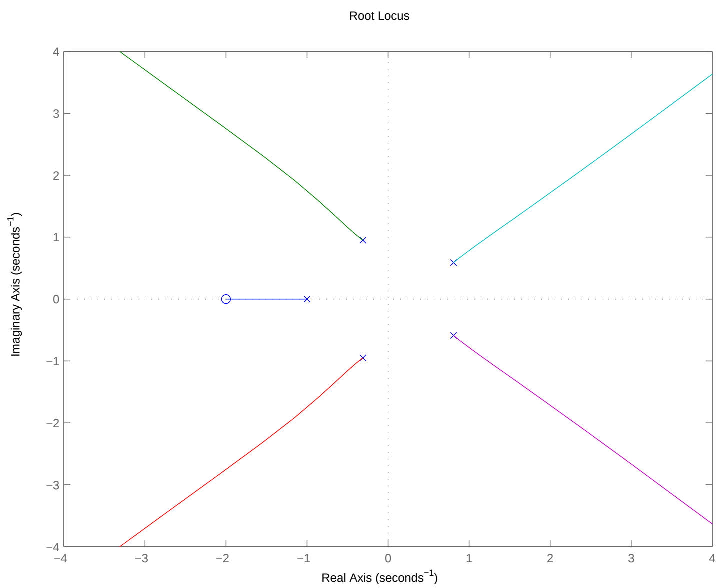
Rule C: $m = 1 \Rightarrow 1$ branch ends at OL zeros.

Rule D: Real locus: $[-2, -1]$

Rule E: $n - m = 3 \Rightarrow \alpha = \dfrac{-1 - 2 * 0.309 \pm j0.9511 + 2 * 0.809 \pm j0.5878 - (-2)}{4} = 0.5$

$\phi_l = \dfrac{180° + 360°(l-1)}{n-m}, l = 1, .., n - m - 1 = \pm 45°, \pm 135°$

Rule F: No $j\omega-$crossing



Problem 2

$$\frac{Y(s)}{R(s)} = \frac{\frac{c+2s}{c+s}}{1+\frac{1}{2s^2}\frac{c+2s}{c+s}}$$
$$= \frac{4s^3+2cs^2}{2s^3+2cs^2+2s+c}$$

Characteristic equation:

$$1+\frac{1}{2s^2}\frac{c+2s}{c+s} = 0 \Rightarrow s^3+2cs^2+2s+c=0$$
$$\Rightarrow (s^3+2s)+c(2s^2+1)=0$$
$$\Rightarrow 1+c\frac{2s^2+1}{s(2s^2+2)}$$
$$\therefore L(s) = \frac{2s^2+1}{s(2s^2+2)}$$

Root Locus:

OP zeros: $\pm\frac{1}{\sqrt{2}}j$

OP poles: $0, \pm j$

Rule A: $n=3, m=2 \Rightarrow$ 3 Branches.

Rule B: $n=3 \Rightarrow$ 3 branches start at OL poles. $n-m=1 \Rightarrow$ 1 branch goes to infinity.

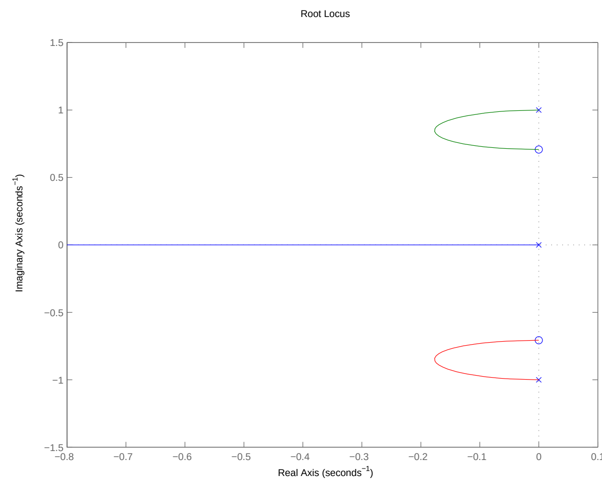
Rule C: $m=2 \Rightarrow$ 2 branches end at OL zeros.

Rule D: Real locus: $(-\infty, 0]$

Rule E: $n-m=1 \Rightarrow \alpha = \frac{\pm j - (\pm\frac{1}{\sqrt{2}}j)}{1} = 0$

$\phi_l = \frac{180^\circ + 360^\circ(l-1)}{n-m}, l = 1, .., n-m-1 = 180^\circ$

Rule F: No $j\omega-$crossing



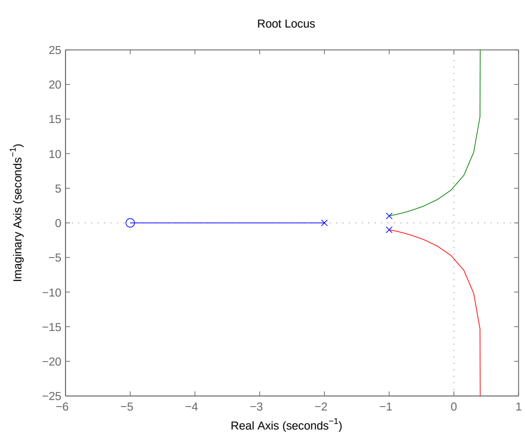
Problem 3

Consider the Evans-style feedback configuration

A complex conjugate root theorem states that if $a + bj$ is a root of a polynomial, P, then its complex conjugate $a + bj$ is also a root of P.

(i) 3 LHP poles and 1 LHP zero
Although, all poles and zeros are in the LHP, it does not necessary mean that all values of gain, K, will keep the system stable. Suppose that the transfer function is strictly proper, the asymptotes should be taken into consideration. If there are $j\omega$-crossings, then some asymptotes may escape into RHP with high gain, causing the system to be unstable.

For example, consider a system with a zero at $s = -5$ and poles at $s = -2, s = -1 \pm j$. Although, it has 3 LHP poles and 1 LHP zero, the root locus plot of this system shows that there are $j\omega$-crossings and the system becomes unstable with high gain.



(ii) 1 RHP pole, 2 LHP poles, and 3 LHP zeros;
Root locus branches start at the poles and end at the zeros. If the number of poles is greater than the number of zeros, then some branch(es) will go to infinity.
In this case, there is one unstable pole (RHP pole). Since there are equal numbers of poles and zeros, all branches will end at the zeros and because all the zeros are in the LHP, a large enough K can stabilize the closed-loop system.

(iii) 5 LHP poles, 4 LHP zeros, 1 RHP zero.
Similarly to the previous problem, all branches will start at the poles and end at the zeros. This case also has equal number of both. However, one of the zeros is in the RHP. Therefore, a large value of K can cause the closed-loop system to be unstable.