WHAT IS THE PROBABILITY OF YOU HAVING A LAB PARTNER WHOSE NAME WOULD APPEAR IMMEDIATELY NEXT TO YOURS IN THE ROSTER?

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ABSTRACT. We propose a problem of pairing up lab partners given class size N and give the solution by computing among all possible pairing-ups the probability of having *at least* one pair of 2 whose names would be adjacent in the roster in alphabetic order. As class size $N \to \infty$, this probability has a limit of $1 - \frac{1}{e}$. Interestingly, this is also the percentage used to define the *time constant* of a stable first order transfer function.

1. Introduction

1.1. **The origin of the problem.** In ECE 486 Control Systems Lab, each student has to find one and only one partner to form a pair. There is also a roster within which everyone's name appears in alphabetic order. The class size is N, which is a positive, even number.

For ten consecutive semesters, we have observed that given a lab of 16 students, there are always at least one group of 2 whose names are adjacent in the roster of alphabetic order. By "adjacent", if A and B form a pair for example, in a class of four students with last name initials A, B, C, D, their names appear as follows in the roster,

$$\begin{matrix} **** \text{ A***} \\ **** \text{ B***} \\ **** \text{ C***} \\ **** \text{ D***} \end{matrix}.$$

Here A, B's names are adjacent in the roster and so are C, D's.

Question 1.2. Is it true that there is *always* at least one pair of 2 students in a class with given size N such that the names of that pair would appear consecutively in the roster in alphabetic order?

Remark 1.1. The answer is no! For example, again consider the case with class size of $N = 4$. Of all the possible pairing-ups, $\{$(A,B)(C,D)$\}$, $\{$(A,C)(B,D)$\}$ and $\{$(A,D)(B,C)$\}$, the second one makes the false case. And in the case of the second pairing-up, a *nonconsecutive case* occurs where no pair exists such that the names of group partners appear immediately next to each other in the roster.

However the class size is definitely a factor. When $N = 2$ it is trivially true since two students have no choices at all and they have to be paired up with each other. So we want to modify Question 1.2 to the one as follows,

Question 1.3. In a class of given size N ($N = 2, 4, \ldots$) students, in the final pairing-up, what is the *probability* of the situation where there are *at least* one lab pair whose names appear consecutively in the roster?

Remark 1.2. All we have to do with the modified problem is to find out all the cases when there is *not* a single pair whose names appear consecutively in alphabetic order in the roster. In the aforementioned example of class of four, out of total number of three combinations, one pairing-up scheme is the nonconsecutive case. So the probability in the case $N = 4$ is $\frac{1}{3}$. This is the *complementary* probability to Question 1.3. We will compute the complementary probability in the following sections before we conclude with the desired probability we are looking for.

Initial investigation was done by [3] writing a recursive computer program to brute-force list all pairing-ups schemes (permutations of numbers 1 through N) and counting the total number of pairs with inner number difference greater than 1 between each group member. The numerical results of class sizes $N = 4$ through 16 are documented as in the table below.

TABLE 1. Tested cases with different class sizes N

Class size N	Total combinations	Nonconsecutive cases	Probability
4	3	1	0.3333333333
6	15	5	0.3333333333
8	105	36	0.3428571429
10	945	329	0.3481481481
12	10395	3655	0.3516113516
14	135135	47844	0.3540459540
16	2027025	721315	0.3576133166

The second column of Table 1 can be computed by

$$\text{Total combinations } (N) := a_N = \frac{\binom{N}{2}\binom{N-2}{2}\cdots\binom{4}{2}\binom{2}{2}}{\left(\frac{N}{2}\right)!}. \tag{1.1}$$

Remark 1.3. The right hand side of a_N in Equation 1.1 can be interpreted as this: In order to form a complete pairing-up of $N/2$ unconstrained pairs, we first choose 2 names out of a set of N names. Then choose another 2 names out of remaining $N - 2$ names. Keep doing this until the last pair out of 2 remaining names. However in the process, the order of choosing does not matter for the final pairing-up, therefore the product $\binom{N}{2}\binom{N-2}{2}\cdots\binom{2}{2}$ is divided by $\left(\frac{N}{2}\right)!$.

If the third column[1] of Table 1 can be computed by an explicit closed form formula of N then the problem is solved.

1.4. **The statement of the original problem.** We are looking for an explicit closed form formula of the sequence in the fourth column of Table 1 in Section 1.1 if N is finite and we are also interested in its limit when $N \to \infty$.

2. Formulation of the problem

Any roster of N names in alphabetic order can be indexed by a set of distinct numbers $\{1, 2, \dots, N\}$. The original problem can be transformed.

Problem 2.1. Given a set of numbers $\{1, 2, \dots, N\}$, N is even. Out of all possible pairing-ups (pair size 2, total number of pairing-ups given by Equation 1.1), how many are without any adjacent number pair $(i, i+1)$, $i \in \{1, 2, \dots, N-1\}$? What is the percentage of all nonconsecutive cases out of all possible pairings and what is the limit of this percentage when $N \to \infty$?

3. Solution to the problem

The solution is given by [2]. We begin with counting how many ways there are if we pick k pairwise disjoint pairs of adjacent numbers $(i, i+1)$ from a set of N numbers.

Lemma 3.1. *Given a set of numbers $\{1, 2, \dots, N\}$, a set of k disjoint pairs of consecutive numbers is $\{(i_j, i_j+1)\}_k$, where $i_j \in \{1, 2, \dots N-1\}$, (i_m, i_m+1) and (i_n, i_n+1) share no common numbers if $m \neq n$, $j, m, n \in \{1, 2, \dots, k\}$. If we denote the total number of such k-pair sets as $b_{N,k}$, then*

$$b_{N,k} = \binom{N-k}{k}. \tag{3.1}$$

Proof. If we see bundled number pair $(i, i+1)$, $i \in \{1, 2, \dots, N-1\}$ as a single "item" as we see single number item i, $i \in \{1, 2, \dots, N\}$, then there are in total k (bundled) plus $N-2k$ (single) items, i.e., $N-k$ items. To pick k disjoint pairs of consecutive numbers out of N numbers is equivalent to choosing k spots from $N-k$ possible options. In the end, it is

$$\binom{N-k}{k}.$$

Hence the Equation 3.1. Q.E.D.[2] □

[1] Experts in Integer Sequence may already notice this column is related to Bessel Polynomials.

[2] There is also an inductive proof for this. See [2].

Using both a_N and $b_{N,k}$, given a set of numbers $\{1, 2, \ldots, N\}$ we can compute the number of pairing-ups with only nonconsecutive numbers paired up.

Lemma 3.2. *Given a_N and $b_{N,k}$ defined as in Equation 1.1 and Equation 3.1 respectively, for a set of N numbers $\{1, 2, \ldots, N\}$, the number of pairing-up cases where only nonconsecutive numbers are paired up is given by*

$$c_N = \sum_{k=0}^{\frac{N}{2}} (-1)^k a_{N-2k} b_{N,k}. \tag{3.2}$$

Proof. A set $\mathbb{S}_N$ is the set of all pairing-ups of numbers $\{1, 2, \ldots, N\}$ containing *at least one* pair of consecutive numbers. If we define $\mathbb{S}_N^i$ as the set of all pairing-ups containing the pair $(i, i+1)$, $i \in \{1, 2, \ldots, N-1\}$, then

$$\mathbb{S}_N = \bigcup_{i=1}^{N-1} \mathbb{S}_N^i.$$

The size of the set $\mathbb{S}_N$, denoted as $|\mathbb{S}_N|$, according to inclusion-exclusion principle is

$$\begin{aligned} |\mathbb{S}_N| = & \sum_{i_1 \in \{1,2,\ldots,N-1\}} |\mathbb{S}_N^{i_1}| - \sum_{i_1, i_2 \in \{1,2,\ldots,N-1\}} |\mathbb{S}_N^{i_1} \cap \mathbb{S}_N^{i_2}| \\ & + \sum_{i_1,i_2,i_3 \in \{1,2,\ldots,N-1\}} |\mathbb{S}_N^{i_1} \cap \mathbb{S}_N^{i_2} \cap \mathbb{S}_N^{i_3}| - \ldots . \end{aligned} \tag{3.3}$$

The size of $\mathbb{S}_N^{i_1}$ is the number of pairing-ups in which $(i_1, i_1 + 1)$ is a pair. Fix i_1, with the remaining $N - 2$ numbers there are a_{N-2} such pairing-ups. The number of choices of i_1 is $N-1$ so the first summation on the right hand side of Equation 3.3 is $a_{N-2}b_{N,1}$.
Consider all pairing-ups containing both (i_1, i_1+1) and (i_2, i_2+1), $i_1 \neq i_2$. The size of $\mathbb{S}_N^{i_1} \cap \mathbb{S}_N^{i_2}$ is a_{N-4} if (i_1, i_1+1) and (i_2, i_2+1) are disjoint or 0 if (i_1, i_1+1) and (i_2, i_2+1) are not disjoint. Therefore the contribution to the second summation on the right hand side of Equation 3.3 comes solely from pairwise disjoint pairs of consecutive numbers, which is $a_{N-4}b_{N,2}$, where $b_{N,2}$ means we have to pick $k = 2$ pairwise disjoint pairs of consecutive numbers $(i, i + 1)$ from $\{1, 2, \ldots, N\}$. This has been previously solved in Lemma 3.1.
Keep this reasoning for all succeeding intersection sets we can reach

$$|\mathbb{S}_N| = \sum_{k=1}^{\frac{N}{2}} (-1)^k a_{N-2k} b_{N,k}.$$

So the total number of pairing-ups where there are no pairs of consecutive numbers is $a_N - |\mathbb{S}_N|$. But

$$\begin{aligned}
a_N - |\mathbb{S}_N| &= a_N - \sum_{k=1}^{\frac{N}{2}} (-1)^k a_{N-2k} b_{N,k} \\
&= (-1)^0 a_{N-2\cdot 0} b_{N,0} - \sum_{k=1}^{\frac{N}{2}} (-1)^k a_{N-2k} b_{N,k} \\
&= \sum_{k=0}^{\frac{N}{2}} (-1)^k a_{N-2k} b_{N,k} \\
&= c_N.
\end{aligned}$$

Q.E.D. $\square$

Theorem 3.3. *Given a_N and c_N defined as in Equation 1.1 and Equation 3.2 respectively, we have*

$$\lim_{N\to\infty} \frac{c_N}{a_N} = \frac{1}{e} \tag{3.4}$$

Proof. Notice that a_N is independent of k. So we can move a_N into each summand of c_N,

(3.5)

$$\begin{aligned}
\frac{c_N}{a_N} &= \frac{\sum_{k=0}^{\frac{N}{2}} (-1)^k a_{N-2k} b_{N,k}}{a_N} \\
&= \sum_{k=0}^{\frac{N}{2}} (-1)^k \frac{a_{N-2k}}{a_N} b_{N,k} \\
&= \sum_{k=0}^{\frac{N}{2}} (-1)^k \frac{\binom{\frac{N}{2}}{2}\binom{\frac{N-2}{2}}{2}\cdots\binom{\frac{N-2k+2}{2}}{2}\binom{N-k}{k}}{\binom{N}{2}\binom{N-2}{2}\cdots\binom{N-2k+2}{2}} \\
&= \sum_{k=0}^{\frac{N}{2}} \frac{(-1)^k}{k!} \times \\
&\quad \frac{\overbrace{N(N-2)\cdots(N-2k+2)}^{k \text{ terms}}\,\overbrace{(N-k)(N-k-1)\ldots(N-2k+1)}^{k \text{ terms}}}{\underbrace{N(N-1)\cdots(N-2k+2)(N-2k+1)}_{2k \text{ terms}}}.
\end{aligned}$$

As $N \to \infty$, the ratio of the two degree $2k$ polynomials is 1 according to L'Hôspital's rule. Therefore

$$\lim_{N\to\infty} \frac{c_N}{a_N} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$
$$= e^x\big|_{x=-1} = \frac{1}{e}.$$

Q.E.D. □

Remark 3.4. If we define $P_N = \frac{c_N}{a_N}$ as the probability of having pairing-ups with only pairs of nonconsecutive numbers, Theorem 3.4 says among all pairing-ups of N students, the probability of having at least one pair of lab partners with theirs names appearing adjacent in a roster in alphabetic order is $1 - \frac{1}{e}$ when $N \to \infty$.

Remark 3.5. For a stable first order transfer function, it takes its time domain step response $t = \tau$ (time constant) to reach $1 - \frac{1}{e} \approx 63.2\%$ of its steady state value.

4. Relations to Bessel Polynomials

$|\mathbb{S}_N|$ in the proof of Lemma 3.2 is the formula for the third column of Table 1. The sequence $\{|\mathbb{S}_N|\}$, $N = 1, 2, \ldots$, can also be generated by Bessel Polynomials $y_n(-1)$ [1].

References

[1] A000806, *Bessel polynomials $y_n(-1)$, online encyclopedia of integer sequences*, URL: `http://oeis.org/A000806`
[2] J. Dover, *What is the probability of no pair's names being adjacent?*, URL (version: 2016-06-02): `http://math.stackexchange.com/q/1809184`
[3] Y. L. Han, *List all possibilities of pair-ups of given numbers 1 through N*, URL (version: 2016-05-31): `http://stackoverflow.com/q/37555296/1489881`
[4] Y. L. Han, *What is the probability of no pair's names being adjacent?*, URL (version: 2016-06-01): `http://math.stackexchange.com/q/1808975`

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