1. Consider a second-order plant with no zeros and with poles at $-\varepsilon \pm j\varepsilon$, where $\varepsilon > 0$ is very small. This system, although stable, doesn’t have good response characteristics. This problem asks you to show, using two different methods, that PD control is a good option for improving the system response. In your answers, appeal to pictures and avoid computations as much as possible. Fit your answers in the space provided.

   a) Using root locus: explain what effect PD control has on the root locus and why it can improve system response in ways that proportional feedback alone cannot.

   Solution: PD control introduces a LHP zero. This zero pulls the root locus into the negative real axis, giving better damping and faster response. With just proportional feedback the poles go vertically away from the real axis, and the response is poor (although the system is stabilized).

   b) Using Bode plots: explain what effect PD control has on the Bode plots and why it can improve system response in ways that proportional feedback alone cannot.

   Solution: The natural frequency $\omega_n$ is very small. Without PD control, the magnitude plot starts high (at $1/(2\varepsilon^2)$) and after passing $\omega_n$ the slope changes to $-2$. We know from Bode’s gain-phase relationship that this is not good for PM, and indeed the phase is close to $-180^\circ$ at $\omega_c$. With PD control, we can change the slope at $\omega_c$ to $-1$ (this is the effect of adding a LHP zero), getting PM closer to $90^\circ$.

2. Decide whether each of the following statements is true or false. If true, justify. If false, explain why and/or modify it to make it true.

   a) The root locus for a second-order plant with a lead controller cannot have points of multiple roots away from the real axis.

   Solution: True. Root locus has 3 branches (2 from plant poles, one from lead pole). If we had a point of multiple roots away from the real axis, then by symmetry of the root locus with respect to the real axis we would also have another point of multiple roots (conjugate to the first one). At each of these two points of multiple roots we would need at least 2 branches to meet, which means we would need at least 4 branches total, but we only have 3. Hence this cannot happen.

   b) Generally, larger crossover frequency $\omega_c$ on the open-loop Bode plot indicates faster closed-loop response (smaller rise time).

   Solution: True. High $\omega_c$ means high closed-loop bandwidth because $\omega_{BW} \in [\omega_c, 2\omega_c]$. Since bandwidth is proportional to natural frequency $\omega_n$, and $\omega_n$ is in turn inversely proportional to the rise time $t_r$, we indeed get small rise time. (The claims are true for prototype second-order response but extend to more general systems by the dominant poles argument).

   c) Generally, larger phase margin indicates smaller settling time (faster decay).

   Solution: False. PM is directly related to the damping ratio $\zeta$, so it gives smaller overshoot $M_p$, not smaller $t_s$. Setting time is related to $\sigma$, the real part of the poles. There is no direct relation between $\zeta$ and $\sigma$ (one quantity can be large while the other is small).

   d) A lead controller $\frac{s + z}{s + p}$, $p > z$ adds phase at all frequencies, with the largest phase added at the frequency $\omega = \frac{1}{2}(z + p)$, halfway between the lead pole and the lead zero.

   Solution: False. The statement becomes true if we replace $\omega = \frac{1}{2}(z + p)$ with $\omega = \sqrt{zp}$, the point halfway between the lead pole and the lead zero on the log scale.
3. Consider the plant \( G(s) = \frac{1}{s^2} \). Suppose that your task is to find a controller of the form \( K \frac{s + z}{s + 2} \) such that the points \(-2 \pm 2j\) become two of the closed-loop poles (the location of the third closed-loop pole is not specified). The value of \( z \) can be either smaller or larger than 2, in other words, your controller can be either a lead or a lag (with pole fixed at \(-2\)). Show that the indicated closed-loop pole locations are impossible to achieve by any choice of \( z \) and \( K \).

**Solution:** Let’s use the root locus and the phase condition. Let \( s = -2 + 2j \) be the test point. The plant has two poles at 0, let’s call them \( p_1 \) and \( p_2 \). The controller has pole at \(-2\), let’s call it \( p_3 \). Let \( \phi_i, i = 1, 2, 3 \) be the angle from pole \( p_i \) to the test point. Let \( \psi \) be the angle from the controller zero (which we’re trying to find) to the test point. The phase condition gives (assuming positive root locus: \( K > 0 \))

\[
\psi - \phi_1 - \phi_2 - \phi_3 = 180^\circ \mod 360^\circ
\]

It is easy to see that \( \phi_1 = \phi_2 = 135^\circ \) and \( \phi_3 = 90^\circ \). Solving for \( \psi \), we get \( \psi = 540^\circ \mod 360^\circ = 180^\circ \). But it is impossible to find a point on the real axis such that the angle from that point to \(-2 + 2j\) equals \(180^\circ\). Hence, the points \(-2 \pm 2j\) are not on the root locus for any choice of \( z \), hence we indeed cannot find values of \( z \) and \( K \) that satisfy the spec. For \( K < 0 \) (negative root locus) we proceed similarly and find that \( \psi \) must be \(0^\circ\), which is again impossible.

This problem can also be solved by a brute-force computation, plugging \(-2 + 2j\) into the closed-loop characteristic equation and showing that it cannot be solved for \( z \) and \( K \). This solution method is not recommended (I didn’t penalize it but it’s self-penalizing because it takes more time and is more prone to computational errors).