1. Consider the transfer function \( G(s) = \frac{1}{s(s^2 + 4s + 8)} \), which already appeared in Problem Sets 5 and 6.

a) Recall (or rederive) the value of \( K \) for which the closed-loop characteristic equation \( 1 + KG(s) \) has roots on the \( j\omega \)-axis.

b) For this value of \( K \), make the Bode plot of \( KG(s) \) using MATLAB and explain how you can confirm the presence of \( j\omega \)-axis closed-loop poles using this plot.

c) Compute the gain and phase margins for \( K = 12 \) using the corresponding Bode plot.

d) Determine the gain \( K \) that gives the phase margin of 60\(^\circ\).

e) Plot the step responses of the closed-loop systems for \( K = 12 \) and the \( K \) you found in part d). Which system has better damping (smaller overshoot)? Why?

Solution:

\[
G(s) = \frac{1}{s(s^2 + 4s + 8)}
\]

a) The critical value for \( K \) is 32 (from HW7), which causes roots of closed-loop system lie on \( s = \pm j\sqrt{8} = \pm j2.8284 \).

b) The attached Bode plot for \( KG(s) \) (\( K = 32 \)) shows that both of GM = PM = 0\( \mid_{\omega = \sqrt{8}} \), which shows that at \( (\omega = \sqrt{8}) \) the \(|KG(j\omega)| = 1 \) and \( \angle KG(j\omega) = -\pi \) which are equivalent to gain and phase conditions of Root Locus.

c) Attached plot shows GM = 8.52 dB (2.67) at \( \omega = \sqrt{8} \) and PM = 45.5\(^\circ\) at \( \omega = 1.45 \).
d) According to the phase plot, for PM = 60°, we need \( \omega_c = 1 \), plugging this in \( |KG(j\omega_c)| = 1 \)
\[ \therefore K = \sqrt{65} \approx 8.1 \]
e) System with \( K = 8.1 \), because larger PM is equivalent to larger \( \zeta \), and larger \( \zeta \) is equivalent to smaller overshoot!

2. Consider the transfer function \( G(s) = \frac{1}{(s - 1)(s^2 + 2s + 5)} \).

a) Derive the values of \( K \) for which the closed-loop characteristic equation \( 1 + KG(s) \) has roots on the \( j\omega \)-axis.

b) For these values of \( K \), make the Bode plots of \( KG(s) \) using MATLAB and explain how you can confirm the presence of \( j\omega \)-axis closed-loop poles using these plots.

c) Compute the gain and phase margins for \( K = 7 \) using the corresponding Bode plot.

d) What is the largest possible phase margin? Determine the gain \( K \) for which it is achieved.

e) The transfer function \( KG(j\omega) \) in this problem has a term of the form \( (j\omega\tau - 1)^{-1} \) (unstable real pole) which has not been considered in class. Performing an analysis similar to the one done in class for a term of the form \( (j\omega\tau + 1)^{-1} \) (stable real pole), explain the contribution of such a term both to the magnitude and to the phase plot.
Solution:

\[ G(s) = \frac{1}{(s - 1)(s^2 + 2s + 5)} \]

a) \( 1 + KG(s) = 0 \Big|_{s=j\omega} \Rightarrow s^3 + s^2 + 3s - 5 + K \Big|_{s=j\omega} = 0 \)

\[ \Rightarrow -j\omega^3 - \omega^2 + 3j\omega - 5 + K = 0 \]

\[ \Rightarrow \begin{cases} 
\omega^3 - 3\omega = 0 & \Rightarrow \omega = 0 \quad \Rightarrow K = 5 \\
\omega^2 + 5 - K = 0 & \Rightarrow \omega = \pm \sqrt{3} \quad \Rightarrow K = 8 
\end{cases} \]

b) According to the attached plots for \( K = 5 \) and \( K = 8 \), we can see that both GM and PM are zero.

c) GM = 1.16 dB, PM = 13.2

d) According to Bode plot, maximum phase (and PM in this case) achieved on \( \omega \approx 1 \) (rad/sec), so we need to choose \( K \) in such a way that this is also equal to \( \omega_c \). The result would be \( K \approx 6.3 \).

e) \((j\omega\tau + 1)^{-1}\) and \((j\omega\tau - 1)^{-1}\) have the same magnitude \( \frac{1}{\sqrt{\omega^2 \tau^2 + 1}} \).

Before the break point \( \omega = \frac{1}{\tau}, (j\omega\tau + 1)^{-1} \approx 1 \). After the break point, \((j\omega\tau + 1)^{-1} \approx (j\omega\tau)^{-1} \). Therefore, its phase changes from 0° to −90°.

The phase of unstable real pole is trickier. Before the break point \( \omega = \frac{1}{\tau}, (j\omega\tau - 1)^{-1} \approx -1 \). After the break point, \((j\omega\tau - 1)^{-1} \approx (j\omega\tau)^{-1} \). Therefore, its phase changes from −180° to −270°.

3. Show that for the transfer function \( KG(s) = \frac{\omega_n^2 \tau^2}{s^2 + 2\zeta \omega_n s} \), the phase margin is independent of \( \omega_n \) and is given by

\[ PM = \tan^{-1} \left( \frac{2\zeta}{\sqrt{4\zeta^4 + 1 - 2\zeta^2}} \right) \]

Solution:

\[ KG(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s} \]
To calculate the phase margin, we first find the gain-crossover-frequency ($\omega_c$):

$$|KG(j\omega)|\bigg|_{\omega=\omega_c} = 1 \Rightarrow \frac{\omega_n^2}{|\omega_c^2 - 2j\omega_n\omega_c\zeta|} = 1$$

$$\Rightarrow \frac{\omega_n^2}{\sqrt{\omega_n^4 + 4\omega_n^2\omega_c^2\zeta^2}} = 1$$

$$\Rightarrow \omega_n^2 = \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2$$

$$\Rightarrow \omega_n^2 = -2\zeta^2\omega_n^2 + \omega_n^2\sqrt{4\zeta^4 + 1}$$

$$\Rightarrow \omega_n^2 = -2\zeta^2\omega_n^2 + \omega_n^2\sqrt{4\zeta^4 + 1 - 2\zeta^2}$$

$$KG(j\omega) = \frac{\omega_n^2}{-\omega_c^2 + 2j\zeta\omega_n\omega_c}$$

$$\angle KG(j\omega) = -\tan^{-1}\frac{2\zeta\omega_n\omega_c}{-\omega_c^2}$$

$$\Rightarrow \theta = \tan^{-1}(x) \iff \pi + \theta = \tan^{-1}(x).$$

Note that $\theta = \tan^{-1}x \iff \pi + \theta = \tan^{-1}(x)$.

4. Consider the system $G(s) = \frac{1}{s(s+1)}$.

a) Design a PD controller that achieves phase margin $\text{PM} \approx 90^\circ$ and closed-loop bandwidth $\omega_{\text{BW}} \approx 10$. Verify that the specs are met (be careful: you will need both open-loop and closed-loop data for this).

b) Can you modify the above design to get $\omega_{\text{BW}} \approx 1$, while maintaining $\text{PM} \approx 90^\circ$? Explain how or why not.

Solution:

$$G(s) = \frac{1}{s(s+1)}$$

a) The bode plot of $G(s)$ (attached) shows that we have a phase margin of $\approx 52^\circ$ (but small $\omega_c$). We want our PD controller to increase $\omega_c$ as well as PM.

$$D(s) = K(\tau s + 1),$$

we choose $1/\tau < 10$ to make sure the gain is high enough at $\omega_c = 10$. Also, we choose $\frac{1}{\tau} < 10$ to make sure that magnitude slope at $\omega_c = 10$ is $-1$.

Let $\tau = 2$ and $K\bigg|_{\omega_c=10} = 1 \Rightarrow K \approx 5 \Rightarrow D(s) = 5(2s + 1)$
b) Achieving $\omega_{BW} = 1$ and $PM = 90^\circ$ is impossible unless we cancel the pole at $s = -1$ (i.e., $D(s) = s + 1$). Because there is a break point at $\omega = 1$ so we can’t maintain slope $= -1$ on that point. Therefore, we cannot make $\omega_{BW} = 1$ and $PM = 90^\circ$ unless we take $D(s) = s + 1$.

5. In class (Thu., Mar 17) we studied the following problem: for the system $G(s) = \frac{1}{s^2}$, design a lead controller that gives $PM \approx 90^\circ$ and $\omega_{BW} \approx 0.5$. This homework problem asks you to check and improve the design given in class.

a) For the controller derived in class:

$$KD(s) = \frac{1}{\frac{s}{16} + 1}$$

compute the PM, open-loop crossover frequency $\omega_c$, and closed-loop bandwidth $\omega_{BW}$. Plot the closed-loop step response. Explain the reasons why this design didn’t fully meet the specs.

b) Improve the design to obtain $PM$ and $\omega_{BW}$ closer to the specs. Does the new closed-loop step response show better damping?

Solution:

$$G(s) = \frac{1}{s^2}$$

a) $$KD(s) = \frac{1}{\frac{s}{16} + 1}$$
Using the bode plot attached, we can see that PM = 63.8° and ωc = 0.606. We can see that PM is far from 90°. For this case, the whole PM should be provided by controller. It means that

$$\sin \phi_m = \frac{p - z}{p + z}$$

where $\phi_m$ is the maximum phase provided by Lead controller and $p$ and $z$ are lead pole and lead zero, respectively. We need either $z \approx 0$ or $\frac{p}{z} \to \infty$.

b) To improve the above design, we need to enlarge $\frac{p}{z}$; an example would be:

$$KD(s) = 2.5 \frac{s + 0.095}{s + 3.8}$$

which improves the PM to 72°.

An “extreme” design is also provided by

$$KD(s) = 5 \frac{s}{s + 10}$$

The bode and time response is attached.
Bode Diagram

$G_m = \infty \text{ dB (at \infty \text{ rad/s})}$, $P_m = 87.1 \text{ deg (at 0.499 rad/s)}$

Step Response

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