Reading: FPE, Sections 5.1, 5.2.

Problems:

1. Consider the plant with transfer function \( L(s) = \frac{1}{s^2 + 2s} \). Under the action of a constant feedback gain \( K \), the closed-loop poles are the roots of the characteristic polynomial \( s^2 + 2s + K \).

   a) Draw the (positive) root locus. (Use the expression for the closed-loop poles in terms of \( K \) obtained via the quadratic formula.)

   b) Consider the settling time spec \( t_s \leq 4 \). Give some value (or range of values) of \( K \) for which the closed-loop system meets this spec. Justify your choice. Show the corresponding pole locations on the root locus.

   c) Consider the rise time spec \( t_r \leq 1 \). Give some value (or range of values) of \( K \) for which the closed-loop system meets this spec. Justify your choice. Show the corresponding pole locations on the root locus.

   d) Consider the overshoot spec \( M_p \leq 0.1 \). Give some value (or range of values) of \( K \) for which the closed-loop system meets this spec. Justify your choice. Show the corresponding pole locations on the root locus.

   e) Suppose that it is desired to place the closed-loop poles at \(-1 \pm j\). Find the value of \( K \) that will achieve this, using the characteristic equation \( s^2 + 2s + K = 0 \) but without using the quadratic formula. (In other words, you should find a way of doing this that would also work for a higher-order example.)

Solution:

a) Root Locus:

\[
\#	ext{ Poles} = n = 2, \#\text{zeros} = m = 0 \\
\Rightarrow \begin{cases} 
\text{# of asymptotes} = n - m = 2 \\
\alpha = \frac{\Sigma p_i - \Sigma z_i}{n - m} = -\frac{2}{2} = -1 \quad \text{(center of asymptotes)} \\
\phi = \frac{(2k + 1)\pi}{n - m} = \pm \frac{\pi}{2} \quad \text{(angle of asymptotes)}
\end{cases}
\]
b) \( t_s \leq 4 \Rightarrow \frac{3}{\sigma} \leq 4 \Rightarrow \sigma \geq 0.75 \)

To satisfy this, we need \( K > 2 \times 0.75 - (0.75)^2 \) or \( K > 0.9375 \) to make sure that both poles are lying on the left of \( (\sigma = 0.75) \) line (dashed).

c) \( t_r \leq 1 \Rightarrow 1.8 \leq 1 \Rightarrow \omega_n \geq 1.8 \)

To make sure the poles are outside the \( \omega_n = 1.8 \) circle, we need

\[ K \geq \omega_n^2 \Rightarrow K \geq 3.24 \]

d) \( M_p \leq 0.1 \)

\[ e^{-\pi \xi} \frac{\pi \xi}{\sqrt{1-\xi^2}} \leq 0.1 \Rightarrow \begin{cases} \frac{\pi \xi}{\sqrt{1-\xi^2}} \geq 2.3 \Rightarrow \xi \geq 0.6 \\ \text{or} \\ \pi \tan \theta \geq 2.3 \Rightarrow \theta \geq 36^\circ \end{cases} \]

but in characteristic eq: \( s^2 + 2s + K \approx s^2 + 2\xi \omega_n s + \omega_n^2 \)

\[ \Rightarrow \omega_n^2 = K \\
\Rightarrow \xi = \sqrt{\frac{1}{K}} \]

\[ \xi \geq 0.6 \Rightarrow \sqrt{\frac{1}{K}} \geq 0.6 \Rightarrow K \leq 2.78 \]

e) Closed loop poles: \( s = -1 \pm j \)

\( \Rightarrow \) characteristic equation: \((s - 1 + j)(s - 1 - j) = s^2 + 2s + 2 \equiv s^2 + 2s + K \Rightarrow K = 2 \)

(**) In general, we can also check the gain condition for root Locus (future lectures).

2. Consider the following transfer functions:

1) \( L(s) = \frac{1}{s(s^2 + 4s + 8)} \)

2) \( L(s) = \frac{s}{(s-1)(s+1)^2} \)

For each one of these, do the following:

a) Mark the zeros and poles on the \( s \)-plane and use Rule 2 from class to plot the real-axis part of the root locus.

b) Use the phase condition from class to test whether or not the point \( s = j \) is on the root locus. If you run into “non-obvious” angles, estimate rather than calculate them, this should be enough.

c) Apply Rules 3 and 4 to determine asymptotes and departure and arrival angles. Plot the root locus branches based on this information.

d) Apply Rule 5 to determine imaginary-axis crossings (if any), and complete the (positive) root locus by using Rule 6 to check for multiple roots.

e) Plot the (positive) root locus using the MATLAB \texttt{rlocus} command.

f) Repeat items a)–e) for the \textit{negative} root locus.

Turn in your MATLAB plots as well as hand sketches of root loci along with all accompanying calculations and explanations.

\textit{Solution:}
1) a) Poles: \( p_1 = 0, p_{2,3} = -2 \pm 2j \)
\( n = 3, m = 0 \Rightarrow 3 \) asymptotes
The negative side of real axis is on loci.

b) Phase condition check
\[
\phi_1 + \phi_2 + \phi_3 = \pi(2k + 1)
\]
\[
\phi_1 = \pi/2, \phi_2 = \tan^{-1}\left(\frac{-1}{2}\right), \phi_3 = \tan^{-1}\left(\frac{3}{2}\right)
\]
\( \Rightarrow \phi_1 + \phi_2 + \phi_3 < \pi \) (because \( \phi_3 + \phi_2 < \pi/2 \))
\[
\therefore s = j \text{ is NOT on loci.}
\]

c) Angle of asymptotes:
\[
\frac{(2k + 1)\pi}{n - m} = \pm \frac{\pi}{3} \text{ and } \pi
\]
Center of asymptotes:
\[
\alpha = \sum p_i - \sum z_i = -\frac{4}{3}
\]
Departure angles:
\[
\phi_{1,dep} = \pi
\]
\[
\phi_{2,dep} = \pi - \frac{3\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}
\]
\[
\phi_{3,dep} = -\phi_{2,dep} = \frac{\pi}{4}
\]
d) $j\omega$-crossings:

$$s(s^2 + 4s + 8) + K\big|_{s=j\omega} = 0$$

$$\Rightarrow -j\omega^3 - 4\omega^2 + 8j\omega + K = 0$$

$$\Rightarrow K = 4\omega^2, \omega^3 - 8\omega = 0 \Rightarrow \omega = 0, \pm \sqrt{8}, \ K = 0, 32$$

Multiple roots:

$$b(s)\frac{da(s)}{ds} - a(s)\frac{db(s)}{ds} = 0$$

$$(3s^2 + 8s + 8) - (s^3 + 4s^2 + 8s) \cdot 0 = s^3 + s^2 - s - 1 = 0$$

$$\Rightarrow s = -1.333 \pm 0.9428 \ (\text{not valid})$$

e)

f) 1-a) Poles: $p_1 = 0, p_{2,3} = -2 \pm 2j$

$n = 3, m = 0 \Rightarrow 3$ asymptotes

The positive side of real axis is on loci.

1-b) Phase condition check

$$\phi_1 + \phi_2 + \phi_3 \overset{?}{=} 2k\pi$$

$$\phi_1 = \pi/2, \phi_2 = \tan^{-1}\left(\frac{-1}{2}\right), \phi_3 = \tan^{-1}\left(\frac{3}{2}\right)$$

$$\Rightarrow \phi_1 + \phi_2 + \phi_3 \neq 2k\pi \ (\text{because } \phi_3 + \phi_2 < \pi/2)$$

$\therefore s = j$ is NOT on loci.
1-c) Angle of asymptotes:
\[
\frac{2k\pi}{n-m} = \pm \frac{2\pi}{3} \text{ and } 0
\]

Center of asymptotes: The same as positive root locus

Departure angles:
\[
\phi_{1,dep} = 0
\]
\[
\phi_{2,dep} = 2\pi - \frac{3\pi}{4} - \frac{\pi}{2} = \frac{3\pi}{4}
\]
\[
\phi_{3,dep} = -\phi_{2,dep} = -\frac{3\pi}{4}
\]

1-d) \(j\omega\) - crossings: From d)-1), \(\omega = 0, \ K = 0\)

Multiple roots: The same as positive root locus.

1-e)

\[
\begin{array}{c}
\text{2) a) Poles: } p_1 = 1, p_{2,3} = -1, \text{ Zeros: } z_1 = 0 \\
n = 3, m = 1 \Rightarrow 2 \text{ asymptotes} \\
The only part of real axis on Loci is } 0 \leq \sigma \leq 1.
\end{array}
\]

\[
\begin{array}{c}
b) \phi_1 + \phi_2 + \phi_3 - \psi_1 = \pi(2k + 1) \\
\phi_1 = \frac{3\pi}{4}, \phi_2 = \phi_3 = \frac{\pi}{4}, \psi_1 = \frac{\pi}{2} \\
\frac{3\pi}{4} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{\pi}{2} = \frac{3\pi}{4} \neq \pi
\end{array}
\]

\[
\therefore s = j \text{ is NOT on loci.}
\]
c) Angle of asymptotes = ±\(\frac{\pi}{2}\)

Center of asymptotes:

\[
\alpha = \frac{\sum p_i - \sum z_i}{n - m} = -\frac{1}{2}
\]

Departure and arrival angles:

\[
\phi_{1,\text{dep}} = \pi
\]

\[
\psi_{1,\text{arr}} = 0
\]

\[
\phi_{2,\text{dep}} = \frac{1}{2}(\pi - 0 - 0) = \frac{\pi}{2}
\]

\[
\phi_{3,\text{dep}} = \frac{1}{2}(\pi - 2\pi) = -\frac{\pi}{2}
\]

\[
\begin{tikzpicture}
\end{tikzpicture}
\]

d) \(j\omega\)-crossings:

\[
(s - 1)(s + 1)^2 + KS_{s=j\omega} = 0
\]

\[
\Rightarrow s^3 + s^2 + (K - 1)s - 1|_{s=j\omega} = 0
\]

\[
\Rightarrow -j\omega^3 - \omega^2 + j\omega(K - 1) - 1 = 0
\]

\[
\Rightarrow \omega(\omega^2 - K + 1) = 0, \quad \omega^2 = -1 \text{ (No solution)}
\]

This can be checked by Routh method.

Multiple roots:

\[
b(s)\frac{da(s)}{ds} - a(s)\frac{db(s)}{ds} = 0
\]

\[
s(3s^2 + 2s - 1) - (s^3 + s^2 - s - 1) = 2s^3 + s^2 + 1 = 0
\]

\[
\Rightarrow s = -1 \text{ (valid)}, \quad s = 0.25 \pm 0.6614j \text{ (not valid)}
\]

e)
f) 2-a) Poles: $p_1 = 1, p_{2,3} = -1$, Zeros: $z_1 = 0$
    $n = 3, m = 1 \Rightarrow 2$ asymptotes
    The part of real axis on Loci is $[-\infty, 0]$ and $[1, \infty]$.
2-b) 
    \[ \phi_1 + \phi_2 + \phi_3 - \psi_1 = 2k\pi \]
    \[ \phi_1 = \frac{3\pi}{4}, \phi_2 = \phi_3 = \frac{\pi}{4}, \psi_1 = \frac{\pi}{2} \]
    \[ \frac{3\pi}{4} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{\pi}{2} = \frac{3\pi}{4} \neq 2k\pi \]
    \[ \therefore \, s = j \text{ is NOT on loci.} \]
2-c) Angle of asymptotes $= 0, \pi$
    Center of asymptotes: The same as positive root locus.
    Departure and arrival angles:
    \[ \phi_{1,dep} = 0 \]
    \[ \psi_{1,arr} = \pi \]
    \[ \phi_{2,dep} = \frac{1}{2} (2\pi + \pi - \pi - \pi) = \pi \]
    \[ \phi_{3,dep} = -\phi_{2,dep} = -\pi \]
2-d) $j - \omega$ crossings: From d)-2), no solution.
    Multiple roots: From d)-2), $s = -1$.

2-e)