**Reading:** FPE, Sections 4.1-4.3 (the material not discussed in class is optional).

**Problems:**

1. Consider the following feedback system, where $K$ is a constant gain and $G(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$:

   ![Feedback System Diagram]

   Using Routh’s criterion, show that for $-1 < K < 3$ the system is stable but for $K \geq 3$ the system is unstable. (This illustrates the destabilizing effect of feedback when the gain is too high.)

   **Solution:**

   $$G_{RY} = \frac{KG}{1 + KG} = \frac{K}{s^3 + 2s^2 + 2s + (K+1)}$$

   Routh array for the denominator of the closed-loop transfer function

   $$s^3: 1 2$$
   $$s^2: 2 K+1$$
   $$s^1: \frac{4-(K+1)}{2} \rightarrow \frac{3-K}{2}$$
   $$s^0: K+1$$

   for stability, we need to make sure that there is no sign change in the first column of the above array. Thus,

   $$\frac{3-K}{2} > 0, \quad K+1 > 0 \quad \Rightarrow -1 < K < 3.$$  

2. The goal of this exercise is to compare sensitivity of open-loop and closed-loop control with respect to errors in the control gain. Consider the DC motor model discussed in class, with no disturbance ($\tau_L = 0$). Let the control gain sensitivity be defined as follows: when the controller gain changes from $K$ to $K + \delta K$ and, as a result, the steady state gain (DC gain) of the overall system changes from $T$ to $T + \delta T$, we define $S_K = \frac{\delta T/T}{\delta K/K}$. (The motor gain $A$ remains fixed here.)

   a) Compute the sensitivity $S_K$ in the open-loop case, starting from the nominal values $K_{ol} = 1/A$ and $T_{ol} = 1$.

   b) Compute the sensitivity $S_K$ for a feedback gain $K_{cl}$, using the approximate formula $\delta T = \frac{dT}{dK}\delta K$ and the fact that the nominal system gain is, as derived in class, $T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}}$.

   Hint: your final answers in a) and b) should be the same as the ones derived in class for sensitivity to errors in the motor gain $A$.

   **Solution:**
a) 

\[ T_{ol} = 1, \quad T_{ol} + \delta T_{ol} = A(K_{ol} + \delta K_{ol}) = A \times \frac{1}{A} + A \times \delta \left( \frac{1}{A} \right) = T_{ol} + A \times \delta \left( \frac{1}{A} \right) \]

\[ \Rightarrow \delta T_{ol} = A \delta \left( \frac{1}{A} \right) = A \delta K_{ol} \]

\[ \Rightarrow S_k = \frac{\delta T_{ol}}{\delta K_{ol}} = \frac{A \delta K_{ol}}{1} = 1 \]

b) 

\[ \frac{\delta T_{cl}}{\delta K_{cl}} = \frac{A}{(1 + AK_{cl})^2} \]

\[ S_{K_{cl}} = \frac{\delta T_{cl}}{\delta K_{cl}} \times \frac{K_{cl}}{T_{cl}} = \frac{A}{(1 + AK_{cl})^2} \times \frac{K_{cl}}{1 + AK_{cl}} = \frac{1}{1 + AK_{cl}} \]

3. Suppose that the DC motor discussed in class is connected in feedback with a PI controller \( k_P + k_I/s \). (This refers to the standard feedback control configuration, where the input to the controller is \( e = r - y = \omega_{\text{ref}} - \omega_m \).) Write down the full transfer function of the closed-loop system in the presence of load/disturbance \( \tau_L \). (For \( k_I = 0 \) this should match what we derived in class for constant feedback gain.) Is it true that by proper choice of gains \( k_P \) and \( k_I \) we can achieve arbitrary pole placement as well as perfect constant reference tracking and constant disturbance rejection in steady state? Justify your answer.

Solution:

\[ \text{Transfer function:} \]

\[ Y = [(k_P + k_I/s)(R - Y) + T_L] \left( \frac{A}{\tau s + 1} \right) \]

which gives:

\[ Y = \frac{A(k_P s + k_I)}{\tau s^2 + (Ak_P + 1)s + Ak_I}R + \frac{As}{\tau s^2 + (Ak_P + 1)s + Ak_I}T_L = G_{RY}R + G_WY_T_L \]

by the proper choice of \( k_P \) and \( k_I \), we can assign the poles of transfer function from input \( (G_{RY}) \) by choosing two parameters for two poles.

Constant reference tracking:

\[ E = 1 - G_{RY} = \frac{\tau s^2 + (Ak_P + 1)s + Ak_I - Ak_P s - Ak_I}{\tau s^2 + (Ak_P + 1)s + Ak_I} = \frac{\tau s^2 + s}{\tau s^2 + (Ak_P + 1)s + Ak_I} \]

\( E \) has a zero DC gain. Also, its final value for constant \( r \left( r(s) = \frac{\alpha}{s} \right) \):

\[ e = \lim_{s \to 0} s \frac{\tau s^2 + s}{\tau s^2 + (k_P A + 1)s + Ak_I} \times \frac{\alpha}{s} = 0 \]
∴ Assuming proper choice for the poles (LHP), PI controller will accomplish perfect tracking.

Constant disturbance rejection: $G_{WY}$ has a zero DC gain. Also, its final value for constant disturbance $W(s) = \frac{2}{s}$ is:

$$y_w(\infty) = \lim_{s \to 0} sG_{WY}(s)W(s) = \lim_{s \to 0} s \frac{As}{\tau s^2 + (Ak_P + 1)s + Ak_I} \times \frac{\beta}{s}$$

∴ Assuming proper choice of poles (LHP), constant disturbances will be rejected perfectly.

4. Consider again the standard feedback configuration like the one in Problem 1, but with $K(s)$ and $G(s)$ unknown transfer functions. Suppose that the transfer function from $R$ to $Y$ is

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

for some $\zeta, \omega_n > 0$.

a) Based on this information, find the forward gain $K(s)G(s)$.

b) Determine the system type and discuss what it implies about the system’s steady-state tracking ability.

**Solution:**

a)

$$\frac{Y}{R} = \frac{K(s)G(s)}{1 + K(s)G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n) + \omega_n^2} = \frac{s(s + 2\zeta\omega_n)}{1 + \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}}$$

∴ $K(s)G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$

b) The system has one pole at the origin. Therefore, the system is Type I system.

$$k_p = \lim_{s \to 0} K(s)G(s) = \lim_{s \to 0} \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} = \infty \Rightarrow \frac{1}{1 + k_p} = 0$$

The system follows constant references (step) without error.

$$k_v = \lim_{s \to 0} sK(s)G(s) = \lim_{s \to 0} s \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} = \frac{\omega_n}{2\zeta} \Rightarrow \frac{1}{k_v} = \frac{2\zeta}{\omega_n}$$

The system follows ramp references with constant error $\frac{2\zeta}{\omega_n}$.

$$k_a = \lim_{s \to 0} s^2K(s)G(s) = \lim_{s \to 0} s^2 \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} = 0 \Rightarrow \frac{1}{k_a} = \infty$$

The system cannot follow parabola references.