Reading Assignment:

**FPE**, Sections 3.3-3.6, 4.1-4.3

Problems:

(unless otherwise noted, you can use a calculator/computer to arrive at numerical answers)

1. Determine whether or not the following polynomials have any RHP roots:
   
   (i) \( s^4 + 10s^3 + 15s^2 + 20s + 1 \)
   
   (ii) \( s^6 + 2s^5 - 3s^4 + s^3 + 2s + 3 \)
   
   (iii) \( s^8 + 4s^7 + s^6 + 2s^5 + 3s^4 - s^3 + 2s + 3 \)
   
   (iv) \( s^4 + 10s^3 + 12s^2 + 20s + 1 \)

   *(Computer use not allowed.)*

2. Consider the following feedback system, where \( K \) is a constant gain and

   \[
   G(s) = \frac{1}{s^3 + 3s^2 + s + 1}:
   \]

   ![Feedback System Diagram]

   Using the Routh–Hurwitz criterion, show that the system is stable for \(-1 < K < 2\) and unstable for \( K \geq 2 \). (This illustrates the destabilizing effect of feedback when the gain is too high.)

3. Consider the following open-loop system, consisting of a stable first-order plant

   \[
   G(s) = \frac{1}{s + p}
   \]

   (thus, \( p > 0 \)) and a scalar-gain controller. There is also a disturbance \( W \) that affects the input to the controller, where \( C \) is a fixed constant.

   ![Open-Loop System Diagram]
(i) Choose the value of controller gain $K$ to guarantee perfect tracking of a constant reference.

(ii) Show that the resulting system is unable to reject constant disturbances and compute the resulting DC gain from $W$ to $Y$.

4. In class, we have introduced the notion of system type and described how it relates to reference tracking capabilities (or lack thereof) of closed-loop feedback. In this problem, we will explore the notion of system type with respect to disturbances.

Consider the following unity-feedback configuration that consists of a controller $K(s)$ and a plant $P(s)$, and also includes an additive disturbance $W$:

Recall that the forward-loop transfer function $K(s)P(s)$ has system type $n$ with respect to reference input if it has a pole of order $n$ at the origin, or, equivalently, if

$$\lim_{s \to 0} [s^n K(s) P(s)] \neq 0.$$ 

Assuming the closed-loop system is stable, this means that $n$ is the lowest degree of a polynomial that cannot be tracked in feedback with zero steady-state error. We will now focus on the control objective of disturbance rejection.

(i) Show that the system type with respect to reference inputs is equal to $n$ whenever

$$\lim_{s \to 0} \frac{1 - T_{r \to y}(s)}{s^n} = \text{const} \neq 0,$$

where $T_{r \to y}$ is the transfer function from the reference $R$ to the output $Y$.

(ii) Write down the transfer function $T_{w \to y}$ from the disturbance input $W$ to the output $Y$.

(iii) We say that the above system has type $k$ with respect to disturbance inputs if

$$\lim_{s \to 0} \frac{T_{w \to y}(s)}{s^k} = \text{const} \neq 0.$$ 

Show that this is the case when $T_{w \to y}$ has a zero of order $k$ at the origin, i.e., if $T_{w \to y}(s) = s^k \frac{A(s)}{B(s)}$, where $A$ and $B$ are polynomials with real coefficients, such that $A(0) \neq 0$ and $B(0) \neq 0$.

(iv) Show that the system of type $k$ with respect to disturbances can achieve perfect steady-state rejection of polynomial disturbances of degree $m < k$, but not when $m \geq k$. 


(v) Finally, consider the example we have discussed in class, where

\[ P(s) = \frac{1}{s^2 + 1} \]

and determine the system type with respect to disturbances for P-control \( K(s) = K_P \), PD-control \( K(s) = K_P + K_D s \), and PID-control \( K(s) = K_P + K_D s + K_I \).