**Problem 1.** Consider the system with open-loop transfer function

\[ G(s) = \frac{s + a}{s^3 + s^2 - s + 2} \]

For \( a = -1 \), sketch the Bode plots of \( G(s) \), determine the values of the feedback gain \( K \) so that the closed-loop system is stable. For these values of \( K \), is the phase margin positive or negative? Repeat the exercise with \( a = 0 \).

**Solution:** The characteristic equation is:

\[ s^3 + s^2 + (K - 1)s + (Ka + 2) = 0 \]

When \( a = -1 \), by Routh-Hurwitz criterion for 3-rd order system, we need \( K - 1 > 0 \), \(-K + 2 > 0\) and \( K - 1 > -K + 2 \). Hence the region of \( K \) for stable system is \( K \in (\frac{3}{2}, 2) \). Set \( K = 1.8 \). It can be checked from Bode plot that the phase margin is negative.

When \( a = 0 \), we need \( K - 1 > 0 \), \( K - 1 > 2 \). Hence the system is stable when \( K > 3 \). Set \( K = 5 \). It can be checked from Bode plot that the phase margin is positive but gain margin is negative.
2. Consider the system
\[
\dot{x} = \begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} u, \quad y = x_2
\]

a) Write down the open-loop characteristic equation. (This involves computing a $3 \times 3$ determinant, which you can do either by hand or in MATLAB using a symbolic variable $s$.) Are all open-loop poles in the LHP?

b) Using the formula given in class, compute the transfer function of this system. (Use the general formula, do not take Laplace transform of individual differential equations. Look up the procedure for inverting a matrix by hand, or use the MATLAB command inv.)

c) Find another state-space realization of the same transfer function, in controller canonical form
\[
\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ -a_3 & -a_2 & -a_1 \\ 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \quad y = (b_3 \ b_2 \ b_1) x
\]

Hint: you should see that, similarly to the $2 \times 2$ case discussed in class, there is a simple relationship between the entries in the above matrices and the coefficients in the transfer function.

Solution:
\[
\dot{x} = \begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} u, \quad y = x_2
\]

(a) Open-loop characteristic equation
\[
\det(sI - A) = \det \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} = s^3 + s^2 - 1 \quad \text{(all coefficients are NOT positive)}
\]

open-loop poles \equiv \text{roots of O-L characteristic equation}
\equiv \{0.75, -0.88 \pm 0.75j\} \implies \text{O-L unstable}

(b)
\[
G(s) = C(sI - A)^{-1}B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} (sI - A)^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
\]
\[
(sI - A)^{-1} = \begin{pmatrix} s & 1 & -2/3 \\ 1 & s + 2 & -1 \\ 0 & 3 & s - 1 \end{pmatrix}^{-1} = \frac{1}{s^3 + s^2 - 1} \begin{pmatrix} s^2 + s + 1 & -s - 1 & \frac{1+2s}{3} \\ -s + 1 & s^2 - s & -3s \\ 3 & s - 2/3 & s^2 + 2s - 1 \end{pmatrix}
\]
\[
\implies G(s) = \frac{2s^2 - 1}{s^3 + s^2 - 1} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}
\]

(c) Controller canonical form: $\{b_1 = 2, b_2 = 0, b_3 = -1, a_1 = 1, a_2 = 0, a_3 = -1\}$
\[
\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \quad y = (-1 \ 0 \ 2) x
3. (post-exam material) Determine (from the controllability matrix) whether or not the following systems are controllable.

a) \( \dot{x} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \)

b) \( \dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 3 \\ -1 & -1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u \)

You can use MATLAB to perform matrix multiplication, but you should know how to do it by hand.

**Solution:**

(a)

\( \dot{x} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \)

Controllability matrix: \( C(A, B) = (B : AB) \)

\( C(A, B) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), \( \det(C(A, B)) = -2 \neq 0 \)

\( \implies C(A, B) \) is full-rank, system is controllable

(b)

\( \dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 3 \\ -1 & -1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u \)

\( C(A, B) = (B : AB : A^2B) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix} \), \( \text{rank}(C(A, B)) = 2 < 3 \) (or \( \det(C(A, B)) = 0 \))

System is NOT controllable

4. (post-exam material) Imagine two cars driving on the same road in the same direction one behind the other and trying to go at the same speed. This situation can be described by the linear system

\[
\begin{align*}
\dot{v}_1 &= -(v_1 - v_2) + f_1 \\
\dot{v}_2 &= -(v_2 - v_1) + f_2
\end{align*}
\]

where, for \( i = 1, 2 \), \( v_i \) is the velocity of car \( i \) and \( f_i \) is an external force (wind, road conditions, etc.) acting on it. The meaning of the above equations is that each car accelerates/decelerates depending on whether it is going slower/faster than the other. Now, suppose that we want to rewrite the above system in the following new coordinates: \( \dot{\bar{v}}_1 := v_1 \) (velocity of car 1) and \( \dot{\bar{v}}_2 := v_2 - v_1 \) (relative velocity of the two cars).

a) Write down the differential equations for \( \dot{\bar{v}}_1, \dot{\bar{v}}_2 \) in terms of \( \dot{v}_1, \dot{v}_2 \) and \( f_1, f_2 \).

b) Write down the original system in state-space form \( \dot{v} = Av + Bf \), the new system in state-space form \( \dot{\bar{v}} = \bar{A}\bar{v} + \bar{B}f \), the coordinate transformation matrix \( T \) from \( (v_1, v_2) \) to \( (\bar{v}_1, \bar{v}_2) \), and verify that the formulas given in class (relating \( A \) with \( \bar{A} \) and \( B \) with \( \bar{B} \) via \( T \)) hold. (Hint: since we have two inputs, \( B \) is a matrix, not a vector.)

**Solution:**

\[
\begin{align*}
\dot{v}_1 &= -(v_1 - v_2) + f_1 = \bar{v}_2 + f_1 \\
\dot{v}_2 &= -(v_2 - v_1) + f_2 = -\bar{v}_2 + f_2
\end{align*}
\]

\[
\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}
\]
(a)

\[ \begin{align*}
\dot{v}_1 &= \dot{v}_1 = \bar{v}_2 + f_1 \\
\dot{v}_2 &= \dot{v}_2 - \dot{v}_1 = -\bar{v}_2 + f_2 - (\bar{v}_2 + f_1) = -2\bar{v}_2 - f_1 + f_2
\end{align*} \]

\[ \Rightarrow \begin{cases} 
\dot{v}_1 = \bar{v}_2 + f_1 \\
\dot{v}_2 = -2\bar{v}_2 + f_2 - f_1
\end{cases} \]

(b)

\[ \begin{align*}
\bar{v}_1 &= v_1 \\
\bar{v}_2 &= v_2 - v_1
\end{align*} \]

\[ \Rightarrow \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \]

\[ \begin{align*}
\dot{v} = T\dot{v} &= TAv + TBu \\
\dot{v} &= TAT^{-1}\bar{v} + TBu \\
&= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{v} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u \\
\dot{v}_1 &= \bar{v}_2 + f_1 \\
\dot{v}_2 &= -2\bar{v}_2 - f_1 + f_2 \]