---

1. All questions on this exam refer to the following system, which may describe a spring-mass system under the action of an external force:

$$\ddot{y} + a\dot{y} + y = u \tag{1}$$

Here y is the horizontal displacement of the spring, u is the force, and $a > 0$ is a damping coefficient. (The spring's mass and stiffness coefficient have been normalized to 1.)

a) Convert equation (1) to the state-space form $\dot{x} = Ax + Bu$, $y = Cx$. Make sure to explicitly define the state variables.

Solution: Consider the following variables, $x = [x_1, x_2]^T$. Suppose, $x_1 = y$ that implies that, $x_2 = \dot{y}$. Using this form we can simply define the above system in a Linear form as follows,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -a \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \tag{2}$$

Thus, the values of $\underline{A}$, $\underline{B}$ and $\underline{C}$ are as follows,

1. $A = \begin{bmatrix} 0 & 1 \\ -1 & -a \end{bmatrix}$
2. $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
3. $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

b) Calculate the transfer function of the system from u to y. The transfer function should be in the form of the prototype 2nd-order response transfer function

$$\frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Identify the damping ratio ζ and natural frequency ω_n.

Solution: Lets take the laplace transform of equation (1) as follows,

$$\mathcal{L}(\ddot{y} + a\dot{y} + y) = \mathcal{L}(u) \tag{3}$$

Let's denote, $\mathcal{L}(y) = Y$ and $\mathcal{L}(u) = U$. Using known results in Laplace transform, we get something as follows,

$$s^2Y + asY + Y = U \tag{4}$$

We thus, get the transfer function as follows,

$$\frac{Y}{U} = \frac{1}{s^2 + as + 1} \tag{5}$$

Comparing this with the known transfer function for a second order-system we get,

$$\omega_n^2 = 1 \Rightarrow \omega_n = 1 \tag{6}$$

$$2\zeta\omega_n = a \Rightarrow \zeta = \frac{a}{2} \tag{7}$$

c) Assume that $0 < a < 2$. Explain how the choice of a in this range affects the rise time, the overshoot, and the settling time (computed, as usual, with reference to the system's step response).
Solution:

1. We know that the rise time, $t_r = \frac{1.8}{\omega_n}$. Since, ω_n is a constant (=1), it doesn't change with the range of a
2. $M_p = e^{-\pi \frac{\zeta}{\sqrt{1-\zeta^2}}}$. Since with an increase in a, ζ goes to 1; Thus, the percentage overshoot, M_p would go to zero. Hence decreasing with increasing in a.
3. $t_s = \frac{4.6}{\zeta\omega_n} = \frac{9.2}{a\omega_n}$.Thus, decreases with an increase in a.

d) (extra credit) Is it true or false that increasing a beyond $a = 2$ improves the settling time? Justify your answer.
Solution: Let's see the exact poles in this problem as follows,

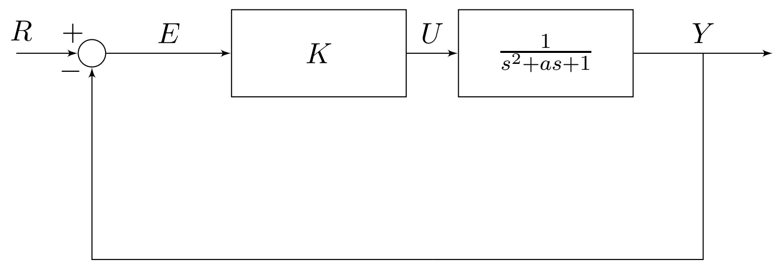
$$z = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{8}$$

For, $a > 2 \Rightarrow \zeta > 1$. The system will start having both real poles beginning with $z = -\omega_n . z = -\omega_n$. Since, one component of this pole starts tending towards the positive side of the plane; It will turn out to be the key factor in deciding the settling time. Since, the real component of this dominant pole is less negative, it won't improve the settling time.
False.

2. Now suppose that, given again the system (1), we want to make the output y track a reference input r by using a feedback controller $u = -Ke$, where $e = r - y$ is the tracking error and K is a constant gain.

a) Draw a block diagram of the closed-loop system. (Note: you are *not* asked to draw an all-integrator diagram; the diagram can have the transfer function of system (1) which you found in Problem 1(b) as a single block.)

Solution:



b) Derive the transfer function of the closed-loop system from r to y.

Solution:

We can use the rule $\frac{\text{forward gain}}{1+\text{loop gain}}$

$$\frac{Y}{R} = \frac{\frac{K}{s^2+as+1}}{1 + \frac{K}{s^2+as+1}} = \frac{K}{s^2 + as + K + 1}$$

c) For what values of K is the closed-loop system stable? Justify your answer.

Solution:

Necessary condition for stability gives $a > 0$ (this is assumed) and $K > -1$. For quadratic polynomials, we know this is also sufficient.

Routh-Hurwitz test can lead to the same result.

d) Suppose the reference is a constant signal (for example, a unit step, $r(t) = \mathbf{1}(t)$). Explain whether the closed-loop system will achieve perfect tracking ($e(t) \to 0$), imperfect tracking ($e(t) \to \text{const} \neq 0$), or no tracking ($e(t) \to \infty$).

Solution:

DC gain $= \frac{K}{K+1} \neq 1 \implies$ imperfect tracking of steps

We can also use Final Value Theorem to calculate $e(\infty)$, which will lead to the same conclusion.

e) Asnwer the same question as in part d) but for the case when the reference is a ramp signal, $r(t) = t\,\mathbf{1}(t)$.

Solution:

There is no pole at $s = 0 \implies$ system is type 0 $\implies$ no tracking of ramps

Same conclusion is reached by applying Final Value Theorem to the transfer function from R to E. But for transfer function from R to Y, FVT gives $y(t) \to \infty$, which doesn't mean anything because $r(t) \to \infty$.

f) Let $a = 1$ and $K = 1$, and let $r(t) = \cos t$. Calculate the steady-state response of the closed-loop system to this input. (As in class, by "steady-state response" we mean the component of the output $y(t)$ that persists after the transients have died down.)

Solution:

Apply the frequency response formula, $y_{ss}(t) = M\cos(\omega t + \varphi)$, where $M = |g(j\omega)|$, $\varphi = \angle g(j\omega)$.

The closed-loop transfer function is

$$g(s) = \frac{K}{s^2 + as + K + 1} = \frac{1}{s^2 + s + 2}$$

$r(t) = \cos t \implies \omega = 1 \implies s = j$

Magnitude: $|g(j)| = \left|\frac{1}{-1+j+2}\right| = \left|\frac{1}{1+j}\right| = \frac{1}{\sqrt{2}}$

Phase: $\angle g(j) = \angle\frac{1}{-1+j+2} = \angle\frac{1}{1+j} = -\angle(1+j) = -\frac{\pi}{4}$

So $y_{ss} = \frac{1}{\sqrt{2}}\cos(t - \frac{\pi}{4})$