Reading: FPE, Sections 6.2, 6.4–6.6, 6.7.1, 6.7.2 (all editions).

Problems: (you can use MATLAB in all problems, but you must explain all steps and justify all answers)

1. Consider the transfer function $G(s) = \dfrac{1}{s(s^2+4s+8)}$, which already appeared in Problem Sets 5 and 6.

a) Recall (or rederive) the value of K for which the closed-loop characteristic equation $1 + KG(s)$ has roots on the $j\omega$-axis.

b) For this value of K, make the Bode plot of $KG(s)$ using MATLAB and explain how you can confirm the presence of $j\omega$-axis closed-loop poles using this plot.

c) Compute the gain and phase margins for $K = 12$ using the corresponding Bode plot.

d) Determine the gain K that gives the phase margin of 60°.

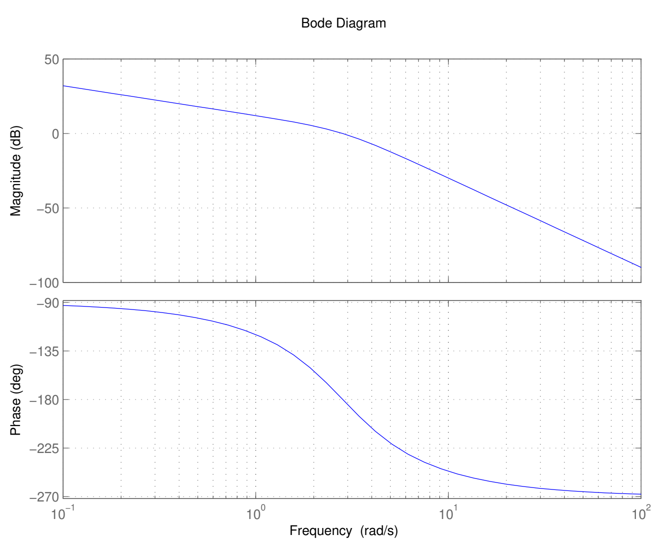
e) Plot the step responses of the closed-loop systems for $K = 12$ and the K you found in part d). Which system has better damping (smaller overshoot)? Why?

Solution:

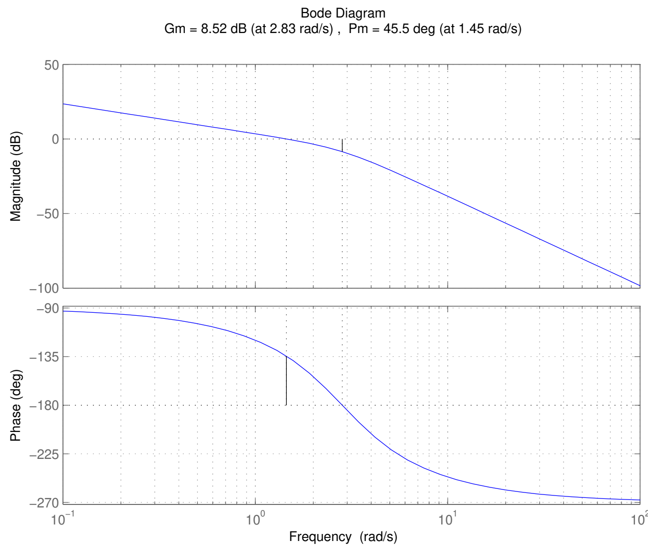
$$G(s) = \frac{1}{s(s^2+4s+8)}$$

a) The critical value for K is 32 (from HW7), which causes roots of closed-loop system lie on $s = \pm j\sqrt{8} = \pm j2.8284$.

b) The attached Bode plot for $KG(s)$ $(K = 32)$ shows that both of GM = PM = $0|_{\omega=\sqrt{8}}$, which shows that at $(\omega = \sqrt{8})$ the $|KG(j\omega)| = 1$ and $\angle KG(j\omega) = -\pi$ which are equivalent to gain and phase conditions of Root Locus.

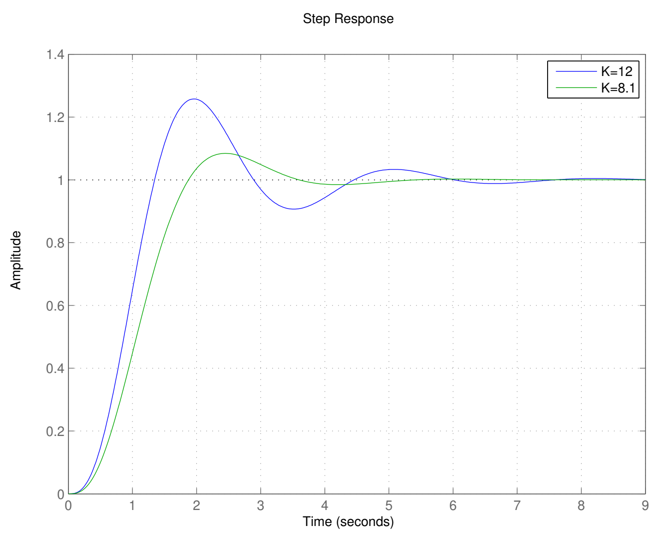


c) Attached plot shows GM = 8.52 dB (2.67) at $\omega = \sqrt{8}$ and PM = 45.5° at $\omega = 1.45$.



d) According to the phase plot, for PM = 60°, we need $\omega_c = 1$, plugging this in $|KG(j\omega_c)| = 1$
$\therefore K = \sqrt{65} \approx 8.1$

e) System with $K = 8.1$, because larger PM is equivalent to larger ζ, and larger ζ is equivalent to smaller overshoot!



2. Consider the transfer function $G(s) = \dfrac{1}{(s-1)(s^2+2s+5)}$.

a) Derive the values of K for which the closed-loop characteristic equation $1 + KG(s)$ has roots on the $j\omega$-axis.

b) For these values of K, make the Bode plots of $KG(s)$ using MATLAB and explain how you can confirm the presence of $j\omega$-axis closed-loop poles using these plots.

c) Compute the gain and phase margins for $K = 7$ using the corresponding Bode plot.

d) What is the largest possible phase margin? Determine the gain K for which it is achieved.

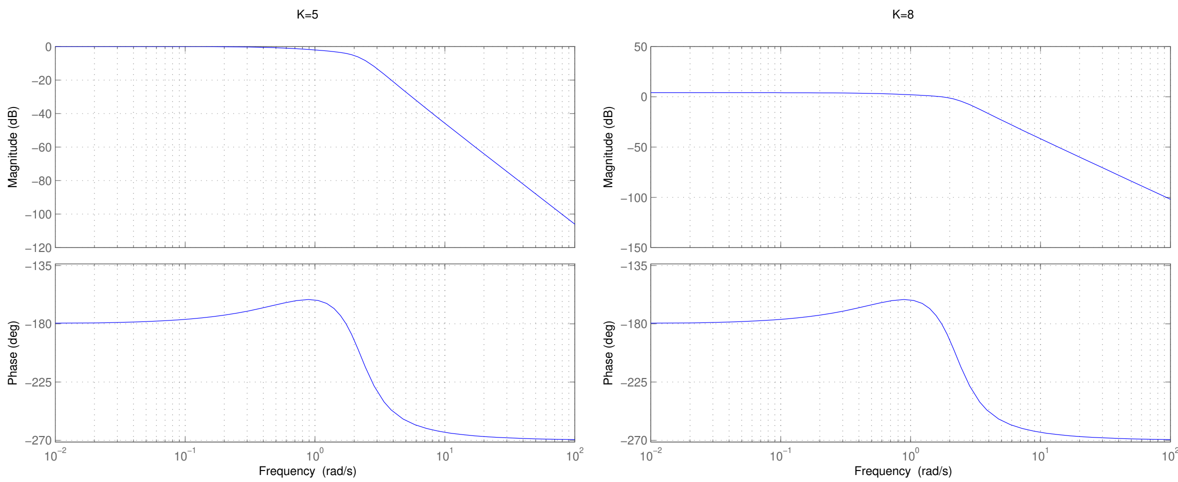
e) The transfer function $KG(j\omega)$ in this problem has a term of the form $(j\omega\tau - 1)^{-1}$ (unstable real pole) which has not been considered in class. Performing an analysis similar to the one done in class for a term of the form $(j\omega\tau + 1)^{-1}$ (stable real pole), explain the contribution of such a term both to the magnitude and to the phase plot.

Solution:

$$G(s) = \frac{1}{(s-1)(s^2+2s+5)}$$

a) $1 + KG(s) = 0|_{s=j\omega} \Rightarrow s^3 + s^2 + 3s - 5 + K\big|_{s=j\omega} = 0$
$\Rightarrow -j\omega^3 - \omega^2 + 3j\omega - 5 + K = 0$
$$\Rightarrow \begin{cases} \omega^3 - 3\omega = 0 \\ \omega^2 + 5 - K = 0 \end{cases} \Rightarrow \begin{matrix} \omega = 0 \\ \omega = \pm\sqrt{3} \end{matrix} \Rightarrow \begin{matrix} K = 5 \\ K = 8 \end{matrix}$$

b) According to the attached plots for $K = 5$ and $K = 8$, we can see that both GM and PM are zero.



c) GM = 1.16 dB, PM = 13.2

d) According to Bode plot, maximum phase (and PM in this case) achieved on $\omega \approx 1$ (rad/sec), so we need to choose K in such a way that this is also equal to ω_c. The result would be $K \approx 6.3$.

e) $(j\omega\tau + 1)^{-1}$ and $(j\omega\tau - 1)^{-1}$ have the same magnitude $\frac{1}{\sqrt{(\omega\tau)^2+1}}$.

Before the break point $\omega = \frac{1}{\tau}$, $(j\omega\tau + 1)^{-1} \approx 1$. After the break point, $(j\omega\tau + 1)^{-1} \approx (j\omega\tau)^{-1}$. Therefore, its phase changes from $0°$ to $-90°$.

The phase of unstable real pole is trickier. Before the break point $\omega = \frac{1}{\tau}$, $(j\omega\tau - 1)^{-1} \approx -1$. After the break point, $(j\omega\tau - 1)^{-1} \approx (j\omega\tau)^{-1}$. Therefore, its phase changes from $-180°$ to $-90°$.

3. Show that for the transfer function $KG(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$, the phase margin is independent of ω_n and is given by

$$PM = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right).$$

Solution:

$$KG(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$$

To calculate the phase margin, we first find the gain-crossover-frequency (ω_c):

$$|KG(j\omega)|\Big|_{\omega=\omega_c} = 1 \Rightarrow \frac{\omega_n^2}{|-\omega_c^2 + 2j\omega_n\omega_c\zeta|} = 1$$
$$\Rightarrow \frac{\omega_n^2}{\sqrt{\omega_c^4 + 4\omega_n^2\omega_c^2\zeta^2}} = 1$$
$$\Rightarrow \omega_n^4 = \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2$$
$$\Rightarrow \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2 - \omega_n^4 = 0$$
$$\Rightarrow \begin{aligned} \omega_c^2 &= -2\zeta^2\omega_n^2 + \omega_n^2\sqrt{4\zeta^4+1} \\ &= \left(\sqrt{4\zeta^4+1} - 2\zeta^2\right)\omega_n^2 \end{aligned}$$

$$KG(j\omega) = \frac{\omega_n^2}{-\omega_c^2 + 2j\zeta\omega_n\omega_c}$$
$$\Rightarrow \begin{aligned} \angle KG(j\omega) &= -\tan^{-1}\frac{2\zeta\omega_n\omega_c}{-\omega_c^2} \\ &= \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}} \end{aligned}$$

Note that $\theta = \tan^{-1} x \iff \pi + \theta = \tan^{-1}(x)$.

4. Consider the system $G(s) = \dfrac{1}{s(s+1)}$.

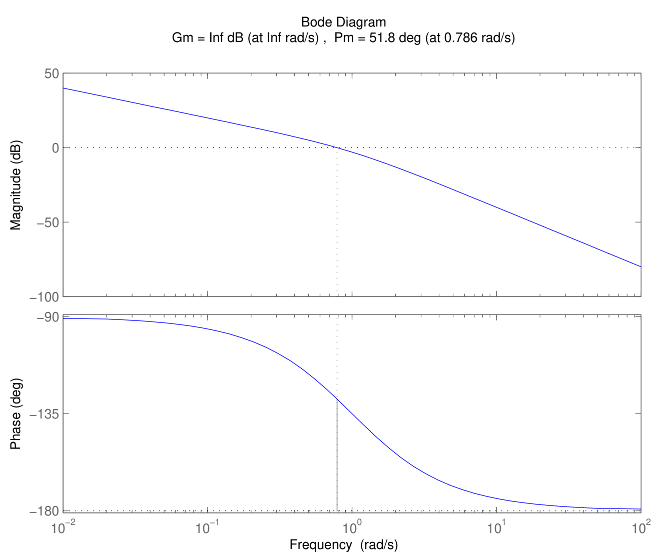
a) Design a PD controller that achieves phase margin PM $\approx 90°$ and closed-loop bandwidth $\omega_{\mathrm{BW}} \approx 10$. Verify that the specs are met (be careful: you will need both open-loop and closed-loop data for this).

b) Can you modify the above design to get $\omega_{\mathrm{BW}} \approx 1$, while maintaining PM $\approx 90°$? Explain how or why not.

Solution:

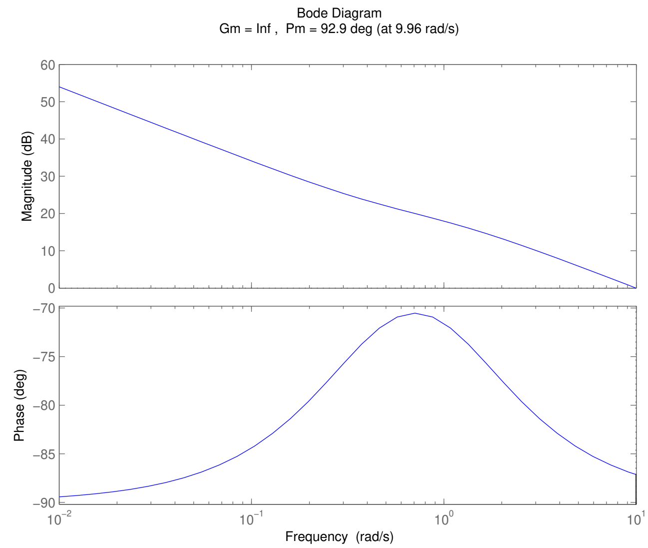
$$G(s) = \frac{1}{s(s+1)}$$

a) The bode plot of $G(s)$ (attached) shows that we have a phase margin of $\approx 52°$ (but small ω_c). We want our PD controller to increase ω_c as well as PM.



$D(s) = K(\tau s + 1)$, we choose $1/\tau \ll 10$ to make sure the gain is high enough at $\omega_c = 10$. Also, we choose $\frac{1}{\tau} < 1$ to make sure that magnitude slope at $\omega_c = 10$ is -1.

Let $\tau = 2$ and $K\left|\frac{2j\omega_c+1}{j\omega_c(j\omega_c+1)}\right|_{\omega_c=10} = 1 \Rightarrow K \approx 5 \Rightarrow D(s) = 5(2s+1)$



b) Achieving $\omega_{\mathrm{BW}} = 1$ and $\mathrm{PM} = 90°$ is impossible unless we cancel the pole at $s = -1$ (*i.e.*, $D(s) = s + 1$). Because there is a break point at $\omega = 1$ so we can't maintain slope $= -1$ on that point. Therefore, we cannot make $\omega_{\mathrm{BW}} = 1$ and $\mathrm{PM} = 90°$ unless we take $D(s) = s + 1$.

5. In class we studied the following problem: for the system $G(s) = \dfrac{1}{s^2}$, design a lead controller that gives $\mathrm{PM} \approx 90°$ and $\omega_{\mathrm{BW}} \approx 0.5$. This homework problem asks you to check and improve the design given in class.

a) For the controller derived in class:

$$KD(s) = \frac{1}{16}\frac{\frac{s}{0.1}+1}{\frac{s}{2}+1}$$

compute the PM, open-loop crossover frequency ω_c, and closed-loop bandwidth ω_{BW}. Plot the closed-loop step response. Explain the reasons why this design didn't fully meet the specs.

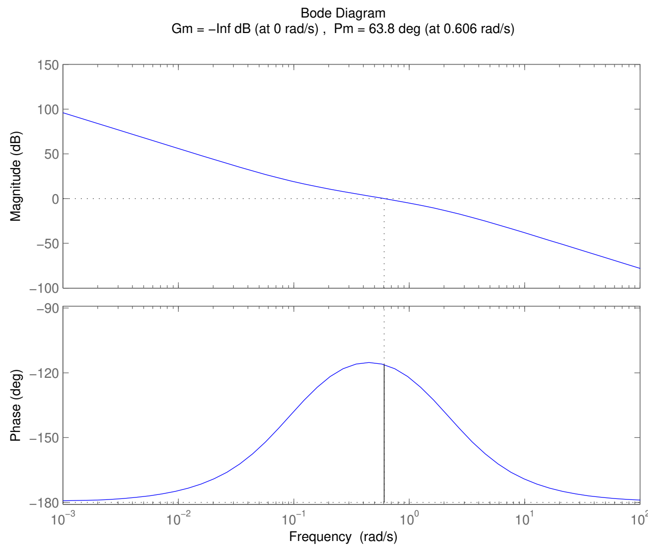
b) Improve the design to obtain PM and ω_{BW} closer to the specs. Does the new closed-loop step response show better damping?

Solution:

$$G(s) = \frac{1}{s^2}$$

a)

$$KD(s) = \frac{1}{16}\frac{\frac{s}{0.1}+1}{\frac{s}{2}+1}$$



Using the bode plot attached, we can see that PM $= 63.8^\circ$ and $\omega_c = 0.606$. We can see that PM is far from 90°. For this case, the whole PM should be provided by controller. It means that

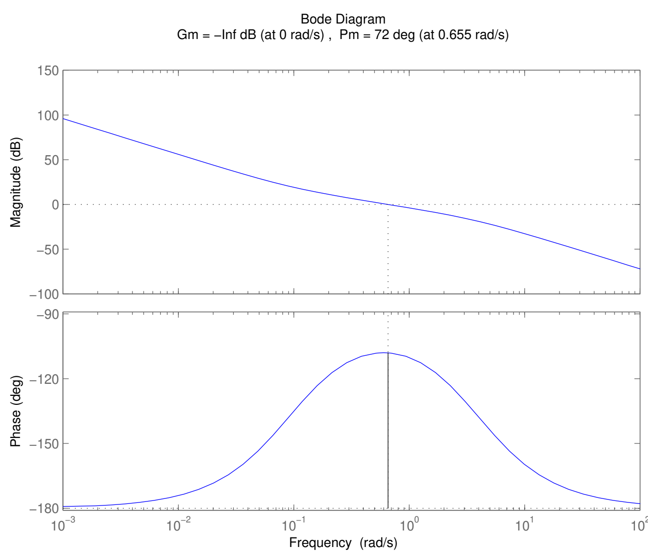
$$\sin\phi_m = \frac{p-z}{p+z}$$

where ϕ_m is the maximum phase provided by lead controller and p and z are lead pole and lead zero, respectively. We need either $z \approx 0$ or $\frac{p}{z} \to \infty$.

b) To improve the above design, we need to enlarge $\frac{p}{z}$, an example would be:

$$KD(s) = 2.5\frac{s+0.095}{s+3.8}$$

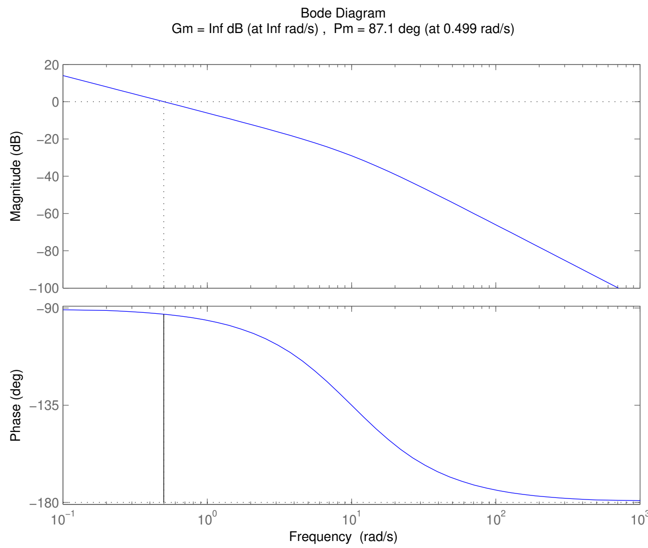
which improves the PM to 72°.



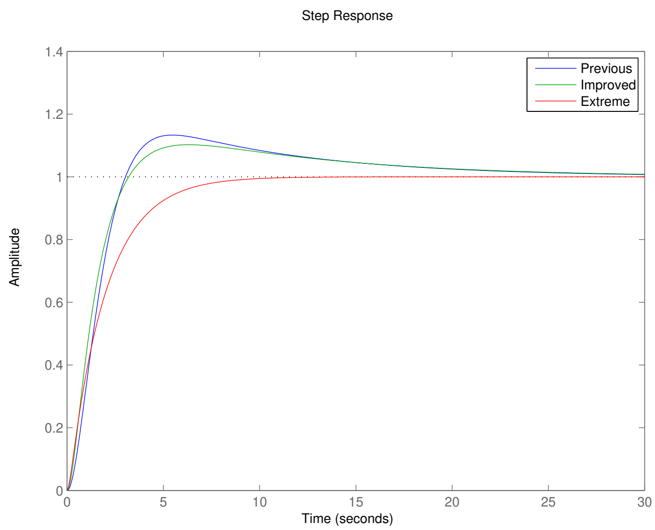
An "extreme" design is also provided by

$$KD(s) = 5\frac{s}{s+10}.$$

The bode and time response is attached.



Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 87.1 deg (at 0.499 rad/s)
Magnitude (dB)
Phase (deg)
Frequency (rad/s)



Step Response
Previous
Improved
Extreme
Amplitude
Time (seconds)