

Reading: FPE (6th edition), Sections 3.1 and 3.2.

Problems:

1. a) Derive *by hand* the Laplace transform of $2 \sin(2t)$. Hint: use Euler's formula. Check your answer in the Laplace transform tables.

b) As we know from class, the (steady-state) frequency response of the system with transfer function $H(s)$ to the signal $A \cos(\omega t)$ is $AM \cos(\omega t + \phi)$, where $H(j\omega) = Me^{j\phi}$. Derive the steady-state response to $A \sin(\omega t)$. (Hint: $H(-j\omega) = Me^{-j\phi}$.)

Solution:

a)

$$\begin{aligned} \mathcal{L}\{\sin(\omega t)\} &= \mathcal{L}\left\{\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right\} = \frac{1}{2j} [\mathcal{L}\{e^{j\omega t}\} - \mathcal{L}\{e^{-j\omega t}\}] \\ &= \frac{1}{2j} \left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right] \\ &= \frac{\omega}{s^2 + \omega^2} \end{aligned}$$

$$\therefore \mathcal{L}\{2 \sin(2t)\} = 2\mathcal{L}\{\sin(2t)\} = \frac{4}{s^2 + 4}$$

b)

$$\begin{aligned} \text{Response to } A \sin(\omega t) &= \frac{A}{2j} [H(j\omega)e^{j\omega t} - H(-j\omega)e^{-j\omega t}] \\ &= \frac{A}{2j} [Me^{j(\omega t + \phi)} - Me^{-j(\omega t + \phi)}] \\ &= AM \times \frac{1}{2j} [e^{j(\omega t + \phi)} - e^{-j(\omega t + \phi)}] \\ &= AM \sin(\omega t + \phi) \end{aligned}$$

2. Consider the system with transfer function $H(s) = \frac{1}{s+2}$. Assume that the initial condition is zero.

a) Compute the response to the input $u(t) = 2 \sin(2t)$ by applying the formula $Y(s) = H(s)U(s)$ with $U(s)$ from problem 1(a), then using partial fractions, and finally Laplace transform tables. You may find the MATLAB command `residue` helpful for checking the results of your partial fractions calculation (but you must derive them by hand and show your work).

b) Compute the response to the same input by using the frequency response formula you obtained in problem 1(b). What is the difference between the answers in part a) and part b), and how is it related to the pole location of the transfer function?

Solution:

a) From problem 1(a): $U(s) = \frac{4}{s^2 + 4}$.

$$\begin{aligned} Y(s) = H(s)U(s) &\Rightarrow Y(s) = \frac{1}{s+2} \frac{4}{s^2+4} = \frac{4}{(s+2)(s^2+4)} \\ &\Rightarrow Y(s) = \frac{A}{s+2} + \frac{Bs+C}{s^2+4} \end{aligned}$$

$$A = (s+2)Y(s)|_{s=-2} = \frac{1}{2}$$

$$Y(s)|_{s=0} = \frac{1}{2} \Rightarrow C = 1$$

$$Y(s)|_{s=-1} = \frac{4}{5} \Rightarrow B = -\frac{1}{2}$$

$$Y(s) = \frac{1/2}{s+2} + \frac{1}{s^2+4} - \frac{1}{2} \frac{s}{s^2+4}$$

Inverse Laplace transform:

$$y(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t).$$

b)

$$H(s) = \frac{1}{s+2} \Rightarrow H(j\omega) = \frac{1}{j\omega+2} \Big|_{\omega=2} = \frac{1}{2\sqrt{2}}e^{-j\pi/4}$$

Note: This calculation holds for steady-state.

$$u(t) = 2\sin(2t) \Rightarrow y(t) = AM\sin(\omega t + \phi)$$

with $A = 2$, $M = \frac{1}{2\sqrt{2}}$, $\omega = 2$ and $\phi = -\pi/4$.

$$\Rightarrow y(t) = \frac{1}{\sqrt{2}}\sin(2t - \pi/4) = \frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t),$$

which is the steady-state part of the response of part (a). The difference, $\frac{1}{2}e^{-2t}$, is the transient response, and its rate of convergence to 0 is determined by the pole of the system.

3. Consider the following transfer functions:

$$H_1(s) = \frac{2}{s^2 - 2s + 4}, \quad H_2(s) = \frac{2s - 3}{s^2 + 4s + 1}.$$

a) Use the final value theorem to compute their DC gains.

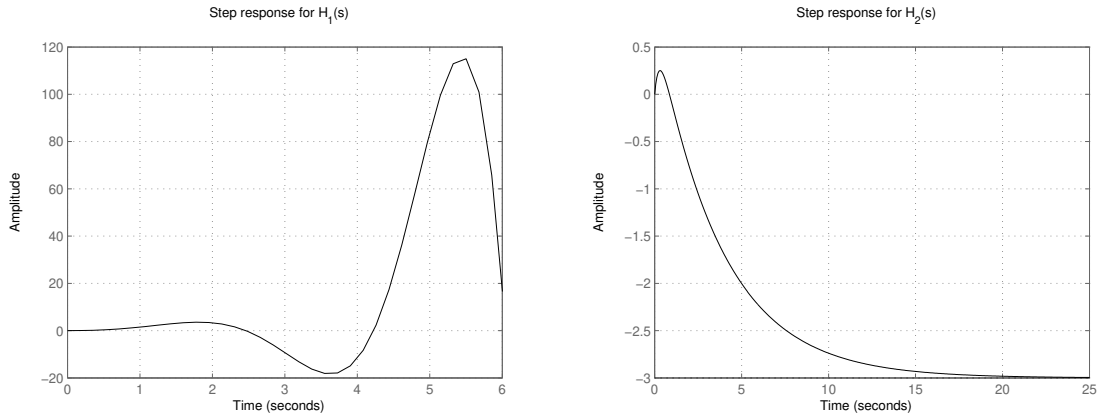
b) Use the MATLAB command `step` to plot their step responses. (You may also find the command `ltiview` convenient to use for such tasks.) Submit your plots.

c) In each case, explain whether the final value theorem gives the right answer and why.

Solution:

- a) DC gain for $H_1(s) = H_1(s)|_{s=0} = \frac{1}{2}$.
 DC gain for $H_2(s) = H_2(s)|_{s=0} = -3$.

b)



- c) As you can see, the DC gain of $H_2(s)$ matches the step response but the DC gain of $H_1(s)$ does not. The reason is because the poles of $H_1(s)$ are not in LHP, hence Final Value Theorem does not apply.

4. Consider the following state-space model (so-called “observer canonical form”)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -a_0 \\ 1 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} u, \quad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- a) Show that its transfer function is $H(s) = \frac{b_1s + b_0}{s^2 + a_1s + a_0}$.

Hint: write out the differential equations, then switch to the s -domain using the differentiation rule for Laplace transforms, and use the resulting equations to solve for $Y(s)$ in terms of $U(s)$.

- b) Build an all-integrator diagram for this system.

Hint: your diagram will be different from the all-integrator diagram given in class for the system in “controller canonical form”, even though the two systems have the same transfer function. But you should be able to see how the two diagrams are related (loosely speaking, they are “mirror images” of one another, with summer junctions and splitters interchanged).

Solution:

- a) Write out the differential equations:

$$\begin{aligned} \Rightarrow \dot{x}_1 &= -a_0x_2 + b_0u \\ \dot{x}_2 &= x_1 - a_1x_2 + b_1u \\ y &= x_2 \end{aligned}$$

Laplace transform:

$$\begin{aligned} \Rightarrow sX_1 &= -a_0X_2 + b_0U \\ sX_2 &= X_1 - a_1X_2 + b_1U \\ Y &= X_2 \end{aligned}$$

$$\begin{aligned} \Rightarrow s^2 X_2 &= sX_1 - a_1 s X_2 + b_1 s U = -a_0 X_2 - a_1 s X_2 + b_0 U + b_1 s U \\ \Rightarrow (s^2 + a_1 s + a_0) X_2 &= (b_1 s + b_0) U \\ \therefore \frac{Y(s)}{U(s)} &= \frac{b_0 + b_1 s}{s^2 + a_1 s + a_0}. \end{aligned}$$

b)

