Reading: FPE (Franklin, Powell, Emami-Naeini, 6th edition), Sections 1.1, 1.2, 2.1-2.4, 7.2, 9.2.1.

Problems: (the first two problems are designed to test your background)

1. For each matrix and/or vector pair given below, compute their product $A \cdot B$ if possible, or explain why it is not possible.

a) $A = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$; b) $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & -3 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; c) $A = \begin{pmatrix} 2 & 3 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 3 & 0 \end{pmatrix}$.

Solution:

a) $A \cdot B = \begin{pmatrix} -15 & 3 \\ -11 & 2 \end{pmatrix}$

b) $A \cdot B$ is not defined because the number of columns of A is not equal to the number of rows of B.

c) $A \cdot B = \begin{pmatrix} 8 & -4 \end{pmatrix}$

2. Compute the magnitude and the phase of the following complex numbers:

a) $1 - 2j$ b) $2 + 3j$ c) $(1 - 2j) \cdot (2 + 3j)$

How do the answers for c) relate to those for a) and b)? State the general rule behind this.

Solution:
There are two basic forms of complex number notation: *polar* and *rectangular*. A complex number $a + bj$ (rectangular form) can be denoted as $\sqrt{a^2 + b^2}\angle \tan^{-1}(b/a)$ (polar form).

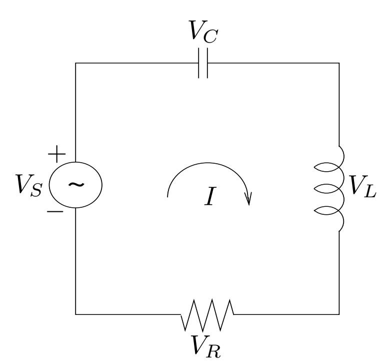
a) $1 - 2j = \sqrt{5}\angle \tan^{-1}(-2) \approx \sqrt{5}\angle - 63.43°$

b) $2 + 3j = \sqrt{13}\angle \tan^{-1}(3/2) \approx \sqrt{13}\angle 56.31°$

c) $(1 - 2j) \cdot (2 + 3j) = \sqrt{5}\sqrt{13}\angle \tan^{-1}(-2) + \tan^{-1}(3/2) \approx \sqrt{65}\angle - 7.12°$

Rule: Let $z_1 = a_1 + b_1 j = m_1\angle\theta_1$ and $z_1 = a_2 + b_2 j = m_2\angle\theta_2$. Then $z_1 \cdot z_2 = m_1 \cdot m_2\angle\theta_1 + \theta_2$.

3. Derive a state-variable model, of the form $\dot{x} = Ax + Bu$, for the following circuit:



Note that you have to decide which variables to take as the states and which one to take as the input. Make sure to declare you choice.

Solution:
Apply KVL on the loop:

$$V_S = V_C + L\frac{dI}{dt} + IR, \; I = C\frac{dV_C}{dt}$$

Define V_C and I as the states and V_S as the input: $x_1 = I$, $x_2 = V_C$ and $u = V_S$.

$$u = x_2 + L\dot{x_1} + Rx_1, \; x_1 = C\dot{x_2}$$

Write the equations in matrix form:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/L \\ 0 \end{pmatrix} u$$

Therefore, $A = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1/L \\ 0 \end{pmatrix}$.

4. Convert each of the following high-order differential equations into the state-variable form:

a) $\ddot{x} + 2\dot{x} - 0.5x = 2u$ b) $x^{(4)} + \ddot{x} + x = u$

Solution:

a) Let $z_1 = x$ and $z_2 = \dot{x}$.

$$\begin{aligned} \dot{z_1} &= z_2 \\ \dot{z_2} &= -2z_2 + 0.5z_1 + 2u \end{aligned}$$

$$\Rightarrow \begin{pmatrix} \dot{z_1} \\ \dot{z_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0.5 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u$$

b) Let $z_1 = x$, $z_2 = \dot{x}$, $z_3 = \ddot{x}$ and $z_4 = x^{(3)}$.

$$\begin{aligned} \dot{z_1} &= z_2 \\ \dot{z_2} &= z_3 \\ \dot{z_3} &= z_4 \\ \dot{z_4} &= -z_3 - z_1 + u \end{aligned}$$

$$\Rightarrow \begin{pmatrix} \dot{z_1} \\ \dot{z_2} \\ \dot{z_3} \\ \dot{z_4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u$$

5. Derive the linearization of the equation $\dot{x} = \cos x$ around the equilibrium point $x = \pi/2$ using the following two methods:

a) Use the linear Taylor approximation $f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$ with $x_0 = \pi/2$ to obtain the linearized equation in the form $(x \dot{-} x_0) = f'(x_0) \cdot (x - x_0)$. Note: $\dot{x}$ and $(x \dot{-} x_0)$ are the same (since x_0 is constant) but we want to express the linearized equation in terms of the *deviation from the equilibrium.*

b) Make the coordinate shift $z = x - \pi/2$, linearize the system $\dot{z} = \cos(z + \pi/2)$ around $z = 0$, and then re-express the resulting equation in terms of x.

Make sure the two results agree. Provide a graphical explanation of why the linear function that you found approximates $\cos x$ well near $x = \pi/2$.

Solution:

a)

$$\dot{x}(= f(x)) \approx f(x_0) + f(x_0)'(x - x_0)$$

Moving the $f(x_0)(= \dot{x_0})$ term to the left hand side:

$$(x \dot{-} x_0) = f(x_0)'(x - x_0)$$

Here $f(x) = \cos x$ with $x_0 = \pi/2$.

$$f(x_0)'(x - x_0) = -\sin(\pi/2)(x - \pi/2) = -(x - \pi/2)$$

$$(x \dot{-} \pi/2) = -(x - \pi/2)$$

b) $\dot{z} = \cos(z + \pi/2) = -\sin(z)$. Here $f(z) = -\sin z$ with $z_0 = 0$. Use the same routine:

$$f(z_0)'(z - z_0) = -\cos(0)(z - 0) = -z$$

$$\dot{z} = -z$$

Replacing $z = x - \pi/2$, we get the same results.