NOTE: You don't need to submit this problem set, it is just to help you prepare for the final exam. Solutions will be posted on the web.

Reading: FPE, Sections 7.6 and 7.10.2 (6th and 5th editions), 7.4 and 7.9.2 (4th edition).

Problems:

1. In class we derived the closed-loop system obtained with dynamic output feedback in $(x, \hat{x})$-coordinates:

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & -BK \\ LC & A - LC - BK \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

and later rewrote it in (x, e)-coordinates. Rewrite the same system in $(\hat{x}, e)$-coordinates.

Solution:

$$\dot{x} = Ax - BK\hat{x} \tag{1}$$

$$\dot{\hat{x}} = (A - BK)\hat{x} + LC(x - \hat{x}) \tag{2}$$

Subtracting equation (1) and (2),

$$\dot{x} - \dot{\hat{x}} = A(x - \hat{x}) - LC(x - \hat{x})$$

Since $e = x - \hat{x}$,

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A - BK & LC \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} \hat{x} \\ e \end{pmatrix}$$

2. Consider the plant transfer function $G(s) = \dfrac{1}{s(s+1)}$.

a) Find any controllable and observable state-space realization of $G(s)$.

b) Stabilize the state-space system from part a) by dynamic output feedback. Select arbitrary controller and observer poles such that the closed-loop system is stable and has reasonable damping (in your judgement).

c) Compute the transfer function of the controller you found in part b). Write it in the form $kD(s)$, where k is a scalar gain (not to be confused with the state feedback gain matrix K) and $D(s)$ is a ratio of monic polynomials (leading coefficients equal 1).

d) Draw the (positive) root locus for $L(s) = D(s)G(s)$ and find on it the locations of the closed-loop poles you chose in part b).

e) Draw the Bode plot for $kD(s)G(s)$ and compute the gain margin and phase margin.

f) Decide whether you're happy with the closed-loop system. If not, go back and improve the design.

Solution:

$$G(s) = \frac{1}{s(s+1)}$$

a) Using Matlab `tf2ss`:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \implies A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & 1 \end{pmatrix},\ D = 0$$

We can check $\mathcal{C}(A, B)$, $\mathcal{O}(C, A)$ and see that the realization is observable and controllable

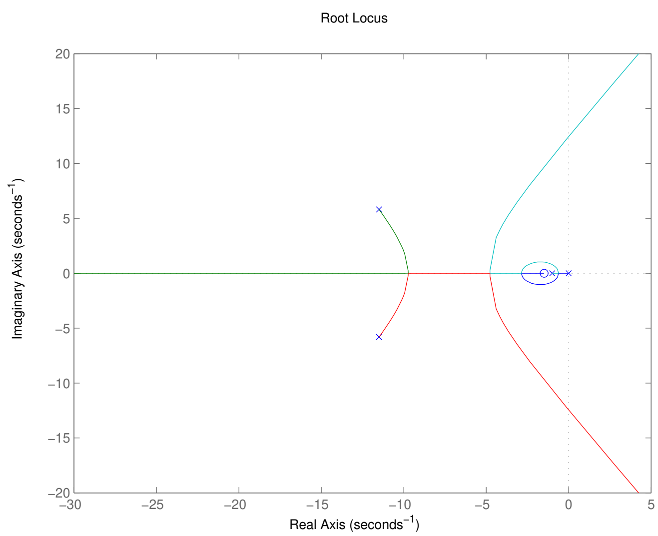
b,c) Assume the closed-loop poles: $-2 \pm j$ and the observer poles : $-10 \pm 2j$

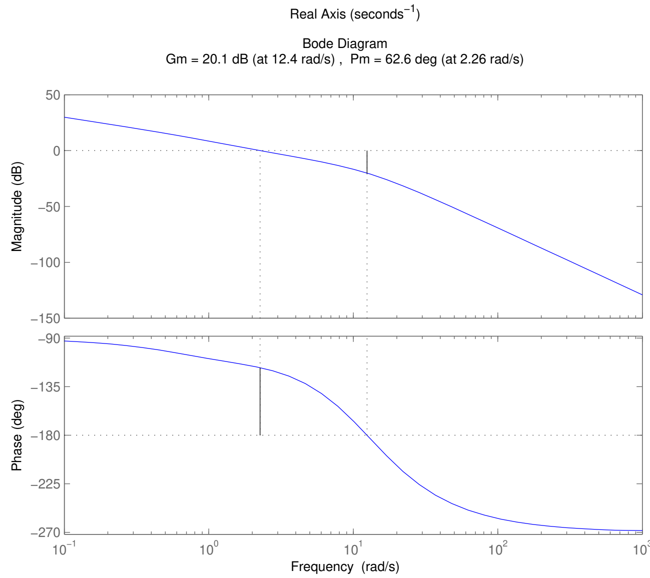
$$\implies \begin{aligned} \mathcal{K} &= \begin{bmatrix} 3 & 5 \end{bmatrix} \\ \mathcal{L} &= \begin{bmatrix} 85 \\ 19 \end{bmatrix} \end{aligned} \implies D_1(s) = K(sI - A + B\mathcal{K} + \mathcal{L}C)^{-1}\mathcal{L}$$

$$D_1(s) = \frac{350s + 520}{s^2 + 23s + 166}$$

$$D_1(s) = kD(s),\ k = 350,\ D(s) = \frac{s + 1.4857}{s^2 + 23s + 166}$$

d,e)





f) It is a good design, stable, and good PM, GM. If we need more bandwidth, we should assign the CL poles farther left in LHP.

3. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= -x_1 + x_2 + u \\ y &= 2x_1 + x_2\end{aligned}$$

and suppose that the control objective is to minimize the performance index $\int_0^\infty [\rho y^2(t) + u^2(t)]dt$, $\rho > 0$.

a) Show graphically the locations of the optimal closed-loop poles as the parameter ρ varies (symmetric root locus).

b) See why in the limit as $\rho \to 0$ ("expensive control" case), the optimal closed-loop poles become mirror images of the open-loop poles across the imaginary axis.

c) See why in the limit as $\rho \to \infty$ ("cheap control" case), one optimal closed-loop pole cancels the open-loop zero and the other moves off to $-\infty$.

Solution:

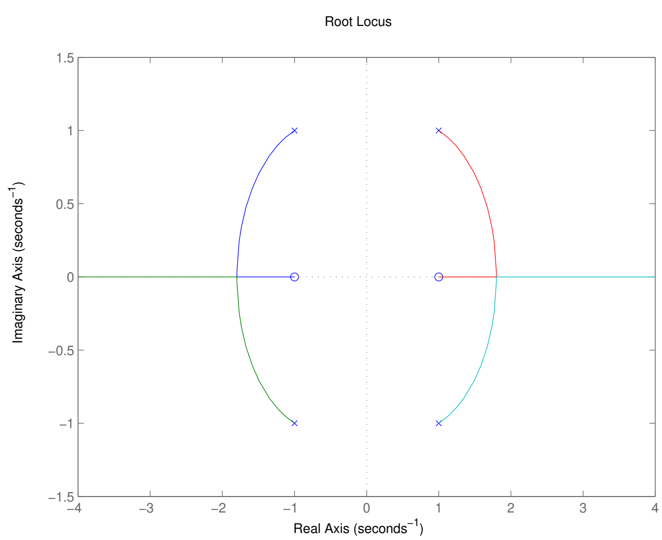
$$\begin{aligned}\dot{x} &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 2 & 1 \end{pmatrix} x\end{aligned}$$

a)

$$G(s) = C(sI - A)^{-1}B = \frac{s+1}{s^2 - 2s + 2}$$

$$G(s)G(-s) = \frac{(s+1)(-s+1)}{(s^2 - 2s + 2)(s^2 + 2s + 2)}$$

The symmetric RL is attached.



b) open-loop poles: $1 \pm j$ from the attached SRL, we see that for every ρ, we assign the LHP pole for the CL system (due to symmetry vs imaginary axis)

For $\rho = 0$, the CL-optimal poles would be $-1 \pm j$ because the OL-poles are on RHP, it would be the mirror images of OL-poles across the imaginary axis (Note: if the OL-poles were on LHP, then CL-poles would be equal to OL poles in this case).

c) For $\rho \to \infty$, one pole moves to zero, the zero is on LHP, so it cancels the pole, and the other one goes to $-\infty$.