Reading: FPE, Sections 7.5, 7.7, 7.8 (6th and 5th editions), 7.3, 7.5, 7.6 (4th edition). Note: some material is developed differently in the book than in class. In these cases, the knowledge of the class approach is mandatory while the knowledge of the book approach is optional.

Problems: (you can use MATLAB to perform necessary matrix computations)

1. Consider the system

$$\dot{x} = \begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} u$$

a) Using the procedure described in class (based on converting to CCF), design a full-state feedback law $u = -Kx$ which places the closed-loop poles at -10 and $-10 \pm 5j$.

b) If the real parts of the desired closed-loop poles were -100 instead of -10, what would happen to the control gains? Give a conceptual answer to this question, without making any extra calculations.

Solution:

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{B} u$$

(a) desired poles: $-10, -10 \pm 5j \implies$ desired characteristic eqn:

$$(s+10)(s+10+5j)(s+10-5j) = s^3 + 30s^2 + 325s + 1250$$

using result of part c of problem 2 HW#10, we can rewrite the above state-equation into

$$\dot{\bar{x}} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}}_{\bar{A}} \bar{x} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\bar{B}} u$$

this can be obtained using similarity transform

$$T = C(\bar{A}, \bar{B})C(A, B)^{-1} = \begin{pmatrix} 0 & -1 & 2/3 \\ 1 & 0 & -1/3 \\ 0 & 0 & +1/3 \end{pmatrix}$$

now

$$\bar{A} - \bar{B}\bar{K} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} \bar{k}_1 & \bar{k}_2 & \bar{k}_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-\bar{k}_1 & -\bar{k}_2 & -1-\bar{k}_3 \end{pmatrix}$$

and

$$\det(sI - \bar{A} - \bar{B}\bar{K}) = s^3 + (1+\bar{k}_3)s^2 + \bar{k}_2 s + (\bar{k}_1 - 1)$$

$$\implies \begin{cases} \bar{k}_3 + 1 = 30 \implies \bar{k}_3 = 29 \\ \bar{k}_2 = 325 \implies \bar{k}_2 = 325 \\ \bar{k}_1 - 1 = 1250 \implies \bar{k}_1 = 1251 \end{cases} \implies \bar{k} = \begin{pmatrix} 1251 & 325 & 29 \end{pmatrix}$$

and state-feedback in original coordinates:

$$\begin{aligned} K = \bar{K}T &= \begin{pmatrix} 1251 & 325 & 29 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2/3 \\ 1 & 0 & -1/3 \\ 0 & 0 & 1/3 \end{pmatrix} \\ &\approx \begin{pmatrix} 325 & -1251 & 735.33 \end{pmatrix} \end{aligned}$$

(b) We can see the new desired characteristic equation:

$$s^3 + 300s^2 + 30025s + 1002500$$

(the coefficients are much larger) and hence $\bar{k}$ (and k) would be scaled up, i.e. control gains would increase.

2. Consider the system

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ b \end{pmatrix} u\,, \qquad y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

a) Derive the transfer function using the formula given in class, keeping b a general constant.

b) Show that for $b = 0$, there is a pole/zero cancellation in the transfer function and loss of controllability in the system.

Solution:

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ b \end{pmatrix} u \quad y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

(a)

$$\begin{aligned} G(s) = C(sI - A)^{-1}B &\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} s+2 & 0 \\ 0 & s-1 \end{pmatrix} \begin{pmatrix} 1 \\ b \end{pmatrix} \frac{1}{(s-1)(s+2)} \\ &= \frac{s+2+b(s-1)}{(s-1)(s+2)} = \frac{1}{s-1} + \frac{b}{s+2} \end{aligned}$$

(b) For $b = 0$,

$$G(s) = \frac{s+2}{(s-1)(s+2)} = \frac{1}{s-1}$$

we can see that there is a pole-zero cancelation. Also it can be checked by

$$\mathcal{C}(A, B) = \begin{pmatrix} 1 & 1 \\ b & -2b \end{pmatrix}$$

and for $b = 0$

$$\text{rank}(\mathcal{C}(\text{A}, \text{B})) = 1$$

3. Determine (from the observability matrix) whether or not the following systems are observable.

a) $\dot{x} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} x, \qquad y = x_2$ b) $\dot{x} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -5 \\ 3 & 3 & -2 \end{pmatrix} x, \qquad y = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} x$

Solution:

(a)

$$\dot{x} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \quad y = x_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

$$\mathcal{O}(C, A) = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}, \quad \det(\mathcal{O}(C, A)) = 0$$
$$\Longrightarrow \text{system unobservable}$$

(b)

$$\dot{x} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -5 \\ 3 & 3 & -2 \end{pmatrix} x, \quad y = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} x$$

$$\mathcal{O}(C, A) = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 6 & -6 \\ -14 & -4 & -14 \end{pmatrix}, \quad \det(\mathcal{O}(C, A)) = 100$$
$$\Longrightarrow \text{system observable}$$

4. For the system

$$\dot{x} = \begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} x, \qquad y = x_2$$

design an observer with observer poles (poles of $A - LC$) placed at -20 and $-20 \pm 2j$. (Follow the procedure described in class, which involves solving the corresponding pole placement problem for an auxiliary system $\dot{x} = Fx + Gu$.)

Solution:

$$\dot{x} = \begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} x, \quad y = x_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x$$

$$\mathcal{O}(C, A) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 2 & 2 & -5/3 \end{pmatrix}, \quad \det \mathcal{O}(C, A) \neq 0 \Longrightarrow \text{system observable}$$

Auxiliary system ("dual" system)

$$F = A^T, \; G = C^T, \; \dot{\bar{x}} = F\bar{x} + Gu, \; \mathcal{C}(F, G) = \mathcal{O}^T(C, A)$$

The desired characteristic equation for

$$F - GK = (s + 20)(s + 20 + 2j)(s + 20 - 2j) = s^3 + 60s^2 + 1204s + 8080$$

CCF for above system:

$$\dot{x}_1 = \bar{F}x_1 + \bar{G}u$$

where

$$\bar{F} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad \bar{G} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T_{F\to\bar{F}} = \mathcal{C}(\bar{F},\bar{G})\mathcal{C}^{-1}(F,G) = \begin{pmatrix} 3 & 0 & 3 \\ 2 & 0 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\bar{F} - \bar{G}\bar{K} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-\bar{k}_1 & -\bar{k}_2 & -1-\bar{k}_3 \end{pmatrix} \stackrel{(*)}{\Longrightarrow} \begin{cases} \bar{k}_1 = 8081 \\ \bar{k}_2 = 1204 \\ \bar{k}_3 = 59 \end{cases}$$

$$\begin{aligned} \bar{K} &= \begin{pmatrix} 8081 & 1204 & 59 \end{pmatrix}, \\ K = \bar{K}T &= \begin{pmatrix} 26769 & 59 & 28032 \end{pmatrix} \end{aligned}$$

and

$$\mathcal{L} = K^T = \begin{pmatrix} 26769 \\ 59 \\ 28032 \end{pmatrix}$$

5. Consider the system

$$\dot{x} = \begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} u, \qquad y = x_2$$

Combine the results of the Problems 1 and 4 to obtain a controller in the form of dynamic output feedback (observer plus estimated state feedback). Write down the state-space model of the controller as well as its transfer function (you can use MATLAB command `ss2tf` to compute the latter).

Solution:

$$\dot{x} = \begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} u; \quad y = x_2$$

from previous homework we know that to place the poles of closed-loop system at $-10, -10 \pm 5j$ we need state feedback $K = \begin{pmatrix} 325 - 1251 & 735.33 \end{pmatrix}$ and $\mathcal{L} = \begin{pmatrix} 26769 \\ 59 \\ 28032 \end{pmatrix}$ to place observer poles at $-20, -20 \pm 2j$; so the state-space model for the controller would be

$$\begin{aligned} \dot{\hat{x}} &= (A - \mathcal{L}C - BK)\hat{x} + \mathcal{L}y \\ u &= -K\hat{x} \end{aligned}$$

$$A - \mathcal{L}C - BK = \begin{pmatrix} -325 & -25519 & -734.66 \\ -651 & 2441 & 1469.66 \\ -975 & -24282 & -2205 \end{pmatrix}$$

using `ss2tf`:

$$\begin{aligned} D(s) &= -K(sI - A + \mathcal{L}C + BK)^{-1}\mathcal{L} \\ &= -\frac{2.92 \times 10^7 s^2 + 5.43 \times 10^7 s + 4.005 \times 10^7}{s^3 + 89s^2 - 5.86 \times 10^7 s - 5.02 \times 10^7} \end{aligned}$$