1A: $2\cos(t) + \sin(t) = (1+\frac{1}{2i})e^{it} + (1-\frac{1}{2i})e^{-it}$

$$\Rightarrow \mathcal{L}\{f_1(t)\} = (1+\frac{1}{2i})\frac{1}{s-i} + (1-\frac{1}{2i})\frac{1}{s+i} = \frac{2s+1}{s^2+1}$$

1B: $\mathcal{L}\{f_2(t)\} = \frac{1}{s+3}$

1C: $\mathcal{L}\{f_1(t)+f_2(t)\} = \frac{2s+1}{s^2+1} + \frac{1}{s+3}$

2A: (i) pole: -7, zero: none, DC gain: $\frac{5}{7}$

free response: the characteristic equation is $s+7=0$

The pole is at $s=-7$

So the free respone is in the form of $y(t) = Ce^{-7t}$

To determine C, we use I.C. we have $Ce^{-7\cdot 0} = y(0)$

$\therefore C = y(0)$

(ii) So the free response is in the form of $y(t) = y(0)\cdot e^{-7t}$

$G_A(s) = \frac{5}{s+7}$ is equivalent to ODE $\dot{y} + 7y = 5u$

suppose we have a particular solution y_p satisfying $\dot{y}_p + 7y_p = 5u$

Then the forced response is in the form: $y(t) = y_p(t) + Ce^{-7t}$

C can be determined using I.C.

$y_p(0) + Ce^{-7\cdot 0} = y(0)$ $\therefore C = y(0) - y_p(0)$

(iii) So the solution is $y(t) = y_p(t) + (y(0) - y_p(0))\cdot e^{-7t}$

(***) step response, assume $y_p(t) = C_0$ for $t \geq 0$ where C_0 is a constant

$\dot{y}_p = 0$, so we need $7C_0 = 5 \Rightarrow C_0 = \frac{5}{7} \Rightarrow y_p(t) = \frac{5}{7}$ is a particular solution.

We have $\dot{Y}_p + 7Y_p = 7\cdot\frac{5}{7} = 5 = 5u$ ($u=1$ for $t\geq 0$)

So $Y_p = \frac{5}{7}$ is a particular solution

The step response is $y(t) = \frac{5}{7} + (y(0) - \frac{5}{7})e^{-7t}$

$= \frac{5}{7} + (0 - \frac{5}{7})e^{-7t}$

$= \frac{5}{7}(1 - e^{-7t})$

(***). response for $u(t) = t$.

~~If~~ First assume $Y_p(t) = C_0$ for $t\geq 0$ where C_0 is a constant.

$\dot{Y}_p = 0$, $\dot{Y}_p + 7Y_p = 7C_0$

The right side of the ODE is $5u = 5t$.

There is no way to make $7C_0 = 5t$ for all t.

So this form doesnot work. we have to try some other form

Now assume $Y_p(t) = C_0 + C_1 t$ where C_0, C_1 are constants.

$\dot{Y}_p = C_1$, $7Y_p = 7C_0 + 7C_1 t$.

$\dot{Y}_p + 7Y_p = C_1 + 7C_0 + 7C_1 t$.

The right side of the ODE is $5u = 5t$.

we need $C_1 + 7C_0 + 7C_1 t = 5t$ for all t.

So we have $\begin{cases} C_1 + 7C_0 = 0 \\ 7C_1 = 5 \end{cases} \Rightarrow \begin{cases} C_1 = \frac{5}{7} \\ C_0 = -\frac{5}{49} \end{cases}$

So we have $Y_p = -\frac{5}{49} + \frac{5}{7}t$, it's straightforward to verify $\dot{Y}_p + 7Y_p = 5t = 5u$

So the forced response is $y(t) = Y_p + ce^{-7t} = -\frac{5}{49} + \frac{5}{7}t + ce^{-7t}$

Let $t=0$, $c - \frac{5}{49} = y(0)$ so $c = y(0) + \frac{5}{49}$,

We have $y(t) = -\frac{5}{49} + \frac{5}{7}t + (y(0) + \frac{5}{49})e^{-7t}$

The pole $s=-7$ is a negative number.

So FVT can be applied!

2B :(*) pole: $s^2+2s-3=0 \Rightarrow s=-3, s=1$

zero: $4s-6=0 \Rightarrow s=\frac{3}{2}$

DC gain: $s=0 \Rightarrow G_p(0)=2$

(**) Two poles: $s=-3$, $s=1$

The free response is $y(t)=ce^{-3t}+\hat{c}e^{t}$

at $t=0$, $\begin{cases} c+\hat{c}=y(0) \\ -3c+\hat{c}=\dot{y}(0) \end{cases} \Rightarrow \begin{cases} c=\dfrac{y(0)-\dot{y}(0)}{4} \\ \hat{c}=\dfrac{3y(0)+\dot{y}(0)}{4} \end{cases}$

(***) The force response is $y(t)=y_p(t)+ce^{-3t}+\hat{c}e^{t}$.

$(c,\hat{c})$ determined by I.Cs $\begin{cases} y(0)=y_p(0)+c+\hat{c} \\ \dot{y}(0)=\dot{y}_p(0)+\hat{c}-3c \end{cases}$

(****) If $u(t)=1$, we look at ODE: $\ddot{y}+2\dot{y}-3y=4\dot{u}-6u$

Assume $y_p=c_0$ for $t\geq 0$, $\ddot{y}_p=\dot{y}_p=0$, we have.

$-3c_0=-6 \Rightarrow c_0=2 \Rightarrow y_p=2$ is a particular solution

step response is: $y(t)=2+ce^{-3t}+\hat{c}e^{t}$

Applying ICs: $\begin{cases} y(0)=2+c+\hat{c} \\ \dot{y}(0)=\hat{c}-3c \end{cases} \Rightarrow \begin{cases} c=\dfrac{y(0)-2-\dot{y}(0)}{4} \\ \hat{c}=\dfrac{3y(0)-6+\dot{y}(0)}{4} \end{cases}$

(*****) If $u(t)=t$, similar to (2A), if we choose $y_p=c_0$, then the left side of the ODE is a constant and the right side is linear in t

It doesn't work! If we chose $y_p=c_0+c_1t$, $\ddot{y}_p=0$, $\dot{y}_p=c_1$

we have $2c_1-3(c_0+c_1t)=4-6t \Rightarrow c_1=2, c_0=0$

So the force response is $2t+ce^{-3t}+\hat{c}e^{t}$ where $(c,\hat{c})$ are solved from ICs.

FVT: $s=1$ is a ~~pole~~ positive pole, FVT can't be applied!

3C (*) poles: $s^2-2s+5=0 \Rightarrow s_1=1+2j,\ s_2=1-2j$

Zero: none, DC gain: 1

(**) free response: $y(t)= ce^{(1+2j)t}+\hat{c}e^{(1-2j)t}$

$$= ce^t(\cos(2t)+j\sin(2t))+\hat{c}e^t(\cos(2t)-j\sin(2t))$$

$$= (c+\hat{c})e^t\cos(2t)+(cj-\hat{c}j)e^t\sin(2t)$$

(***) forced response: $y(t)=y_p(t)+ce^{(1+2j)t}+\hat{c}e^{(1-2j)t}$.

$(c,\hat{c})$ are determined from I.C.s. $\begin{cases} y(0)=y_p(0)+c+\hat{c} \\ \dot{y}(0)=\dot{y}_p(0)+c(1+2j)+\hat{c}(1-2j) \end{cases}$

$$\Rightarrow \begin{cases} c+\hat{c}=y(0)-y_p(0) \\ 2(cj-\hat{c}j)+c+\hat{c}=2(cj-\hat{c}j)+y(0)-y_p(0)=\dot{y}(0)-\dot{y}_p(0) \end{cases}$$

$$\Rightarrow \begin{cases} c+\hat{c}=y(0)-y_p(0) \\ cj-\hat{c}j=\dfrac{\dot{y}(0)-\dot{y}_p(0)-(y(0)-y_p(0))}{2} \end{cases}$$

$$\Rightarrow y(t)=y_p(t)+(y(0)-y_p(0))\cdot e^t\cos(2t)+\frac{\dot{y}(0)-\dot{y}_p(0)-(y(0)-y_p(0))}{2}e^t\sin(2t)$$

(****) If $u(t)=1$, we look at the ODE $\ddot{y}-2\dot{y}+5y=5u$

assume $y_p=C_0$ for $t\geq 0$, $\ddot{y}_p=\dot{y}_p=0 \Rightarrow 5C_0=5 \Rightarrow C_0=1$

$$y(t)=1+(y(0)-1)\cdot e^t\cos(2t)+\frac{\dot{y}(0)-(y(0)-1)}{2}\cdot e^t\sin(2t)$$

(*****) If $u(t)=t$, assume $y_p=C_0+C_1t \Rightarrow -2C_1+5C_0+5C_1t=5t \Rightarrow C_1=1, C_0=\frac{2}{5}$

$$y(t)=\frac{2}{5}+t+(y(0)-\frac{2}{5})e^t\cos(2t)+\frac{\dot{y}(0)-1-(y(0)-\frac{2}{5})}{2}\cdot e^t\sin(2t)$$

(******) The real part of (s_1,s_2) are $1>0$

FVT can't be applied!