ECE 486: Control Systems

Lecture 2C: Laplace Transform & Transfer Function

Key Takeaways

Frequency domain tools can be used to characterize systems modeled by linear ODEs with constant coefficients.

This lecture introduces:

- Laplace Transform
- Transfer functions

Laplace Transforms and the Transfer Function

Reminder: the *two-sided* Laplace transform of a function $f(t)$ is

$$F(s) = \int_{-\infty}^{\infty} f(\tau) e^{-s\tau} \mathrm{d}\tau, \qquad s \in \mathbb{C}$$

time domain	frequency domain
$u(t)$	$U(s)$
$h(t)$	$H(s)$
$y(t)$	$Y(s)$

convolution in time domain $\longleftrightarrow$ multiplication in frequency domain

$$y(t) = h(t) \star u(t) \quad \longleftrightarrow \quad Y(s) = H(s)U(s)$$

The Laplace transform of the impulse response

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \mathrm{d}\tau$$

is called the transfer function of the system.

Laplace Transforms and the Transfer Function

$$Y(s) = H(s)U(s), \qquad \text{where } H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau$$

Limits of integration:

- We only deal with *causal* systems — output at time t is not affected by inputs at future times $t' > t$
- If the system is causal, then $h(t) = 0$ for $t < 0$ — $h(t)$ is the response at time t to a unit impulse at time 0
- We will take all other possible inputs (not just impulses) to be 0 for $t < 0$, and work with *one-sided* Laplace transforms:

$$y(t) = \int_{0}^{\infty} u(\tau)h(t-\tau)\mathrm{d}\tau$$

$$H(s) = \int_{0}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau$$

Laplace Transforms

(see FPE, Appendix A)

One-sided (or unilateral) Laplace transform:

$$\mathscr{L}\{f(t)\} \equiv F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t \qquad \text{(really, from } 0^-\text{)}$$

— for simple functions f, can compute $\mathscr{L}f$ by hand.

Example: unit step

$$f(t) = 1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathscr{L}\{1(t)\} = \int_0^\infty e^{-st}\mathrm{d}t = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \qquad \text{(pole at } s = 0\text{)}$$

— this is valid provided $\mathrm{Re}(s) > 0$, so that $e^{-st} \xrightarrow{t\to+\infty} 0$.

Laplace Transforms

Example: $f(t) = \cos t$

$$\mathscr{L}\{\cos t\} = \mathscr{L}\left\{\frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}\right\} \qquad \text{(Euler's formula)}$$

$$= \frac{1}{2}\mathscr{L}\{e^{jt}\} + \frac{1}{2}\mathscr{L}\{e^{-jt}\} \qquad \text{(linearity)}$$

$$\mathscr{L}\{e^{jt}\} = \int_0^\infty e^{jt}e^{-st}\mathrm{d}t = \int_0^\infty e^{(j-s)t}\mathrm{d}t = \frac{1}{j-s}e^{(j-s)t}\Big|_0^\infty$$

$$= -\frac{1}{j-s} \qquad \text{(pole at } s = j\text{)}$$

$$\mathscr{L}\{e^{-jt}\} = \int_0^\infty e^{-jt}e^{-st}\mathrm{d}t = \int_0^\infty e^{-(j+s)t}\mathrm{d}t = -\frac{1}{j+s}e^{-(j+s)t}\Big|_0^\infty$$

$$= \frac{1}{j+s} \qquad \text{(pole at } s = -j\text{)}$$

— in both cases, require $\mathrm{Re}(s) > 0$, i.e., s must lie in the right half-plane (RHP)

Laplace Transforms

Example: $f(t) = \cos t$

$$\begin{aligned}
\mathscr{L}\{\cos t\} &= \frac{1}{2}\mathscr{L}\{e^{jt}\} + \frac{1}{2}\mathscr{L}\{e^{-jt}\} \\
&= \frac{1}{2}\left(-\frac{1}{j-s} + \frac{1}{j+s}\right) \\
&= \frac{1}{2}\left(\frac{-\cancel{j} - s + \cancel{j} - s}{(j-s)(j+s)}\right) \\
&= \frac{1}{2}\left(\frac{-2s}{-1 + \cancel{js} - \cancel{js} - s^2}\right) \\
&= \frac{s}{s^2+1} \qquad \text{(poles at } s = \pm j\text{)}
\end{aligned}$$

for $\mathrm{Re}(s) > 0$

Transfer Function

Convolution: $\mathscr{L}\{f \star g\} = \mathscr{L}\{f\}\mathscr{L}\{g\}$
(useful because $Y(s) = H(s)U(s)$)

Example: $\dot{y} = -y + u \qquad y(0) = 0$

Compute the response for $u(t) = \cos t$

$\mathscr{L}\{\dot{y}\} = \int_0^\infty \dot{y}e^{-st}\mathrm{d}t = \int_0^\infty e^{-st}\mathrm{d}y = ye^{-st}\Big|_0^\infty - \int_0^\infty y\mathrm{d}e^{-st}$
The first term is 0 since $y(0) = 0$ and the real part of s is positive. The second term is $s\int_0^\infty ye^{-st}\mathrm{d}t$, which is $sY(s)$.

We have $\mathscr{L}\{\dot{y}\} = -\mathscr{L}\{y\} + \mathscr{L}\{u\}$ which leads to

$$sY(s) = -Y(s) + U(s) \Longrightarrow Y(s) = \frac{1}{s+1}U(s)$$

We already know $U(s) = \frac{s}{s^2+1}$

$$\Longrightarrow Y(s) = H(s)U(s) = \frac{s}{(s+1)(s^2+1)}$$

$$y(t) = \mathscr{L}^{-1}\{Y\}$$

Linear ODEs with Constant Coefficients

- An n^{th} order linear ODE with constant coefficients

$$a_n y^{[n]}(t) + a_{n-1} y^{[n-1]}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = b_m u^{[m]}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t)$$

$$\text{ICs: } y(0) = y_0;\ \dot{y}(0) = \dot{y}_0;\ \ldots;\ y^{[n-1]}(0) = y_0^{[n-1]}$$

- Proper if m≤n and strictly proper if m<n
- Linear models often arise by approximating a nonlinear model. This step is called linearization and will be covered later in the course.
- Transfer function representation:

$$G(s) := \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Example

ODE:

$$6\ddot{y}(t) + 9\dot{y} + 2y = 4\dot{u} + 8u$$

TF:

$$G(s) = \frac{4s+8}{6s^2+9s+2}$$

Matlab:

```
>> G=tf([4 8],[6 9 2])
G =
      4 s + 8
  ---------------
  6 s^2 + 9 s + 2
Continuous-time transfer function.
>> [num,den]=tfdata(G);
>> num{1}
ans =
     0     4     8
>> den{1}
ans =
     6     9     2
```