ECE 486: Control Systems

Lecture 16C: Phase Margin

Key Takeaways

This lecture discusses another safety factor called the phase margin to account for model uncertainty and time delays.

- The phase margin is the amount of allowable variation in the phase of the plant before the closed-loop becomes unstable.
- As a rule of thumb, the closed-loop should remain stable for phase variations in the range $\pm 45^o$.

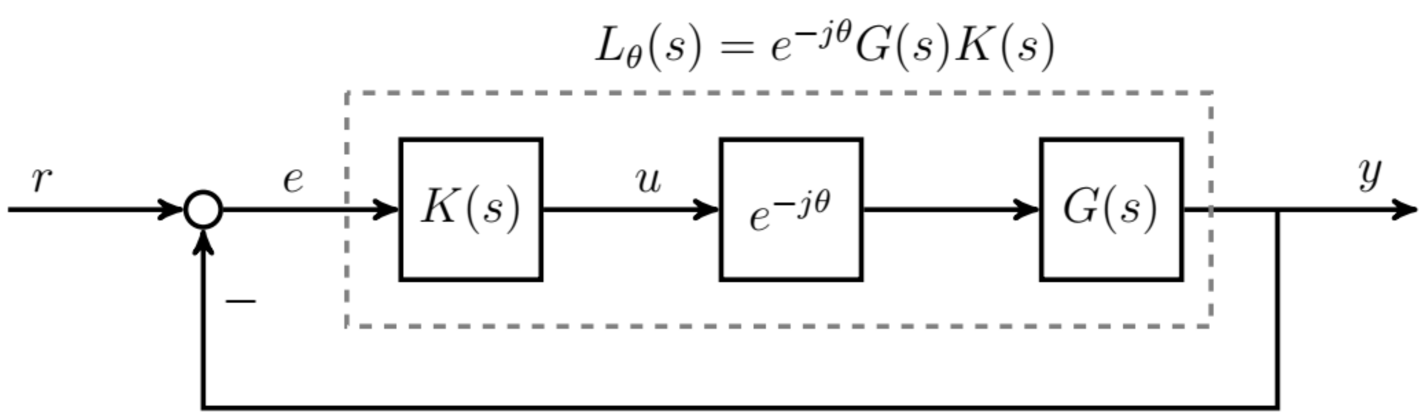
It is shown that a phase variation $\theta_0 > 0$ causes a closed-loop pole at $s = j\omega_0$ if and only if $e^{-j\theta_0} L(j\omega_0) = -1$.

This can be used to determine phase margins from a Bode plot of the loop transfer function $L(s)$.

Phase Margin

The phase margin is another "safety factors" developed to account for the mismatch between the design model and the dynamics of the real system.

- Design model G(s) might differ from real dynamics by a phase $\theta>0$. Phase variations can also occur due to time delays.
- Assume the closed-loop is stable with the nominal model ($\theta=0$).
- The closed-loop may become unstable as θ is varied away from $\theta=0$.
- The phase margin measures the variation before instability occurs.

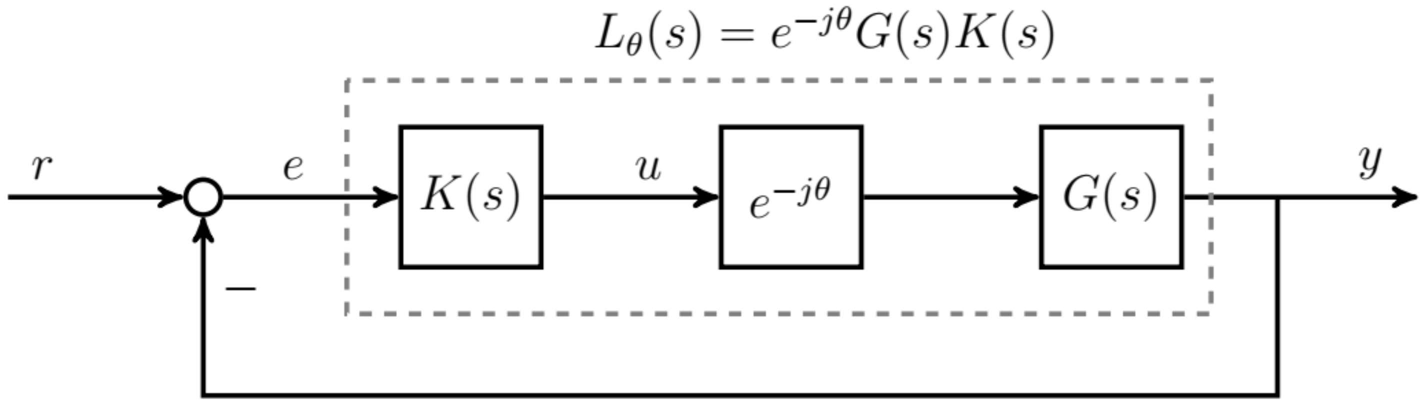


Phase Margin

Definition: The **phase margin** is an upper limit $\bar{\theta} > 0$ such that:

1. the closed-loop is stable for all phase variations θ in the range $-\bar{\theta} < \theta < \bar{\theta}$, and
2. the closed-loop is unstable for $\theta = \bar{\theta}$ (if $\bar{\theta} < \infty$)

As a rule of thumb, the closed-loop should remain stable for phase variations of at least ±45º.



Critical Phases

The closed-loop poles move continuously in the complex plane as the phase θ is varied away from 0.

Critical phases are values of θ for which the closed-loop poles are on the imaginary axis. These phases mark the transition between stable (LHP) and unstable (RHP).

- A critical phase θ_0 causes a closed-loop pole at $s = \pm j\omega_0$.
- A critical phase θ_0 causes $1 + e^{-j\theta_0} L(j\omega_0) = 0$.

A critical phase θ_0 >0 cause a closed-loop pole at $s = \pm j\omega_0$ if and only if $e^{-j\theta_0} L(j\omega_0) = -1$.

Connection to Bode Plots

A critical phase θ_0 >0 cause a closed-loop pole at $s = \pm j\omega_0$ if and only if $e^{-j\theta_0}L(j\omega_0) = -1$. Identify on a Bode plot by:

1. Find frequencies where $|L(\omega_0)| = 1 = 0dB$. These are called *gain crossing or phase margin frequencies.*
2. The critical phase satisfies $-\theta_0 + \angle L(j\omega_0) = -180^o$. Thus the critical phases are $\theta_0 = \angle L(j\omega_0) + 180^o$.
3. This critical phase causes the closed-loop poles at $s = \pm j\omega_0$.

The phase margin $\bar{\theta}$ corresponds to the smallest critical phase (in magnitude).

The Matlab function `allmargin` computes all critical phases and corresponding gain crossing frequencies.

Example

Consider the feedback system with:

$$G(s) = \frac{1}{s^3+2s^2+3s+1} \text{ and } K(s) = 2.$$

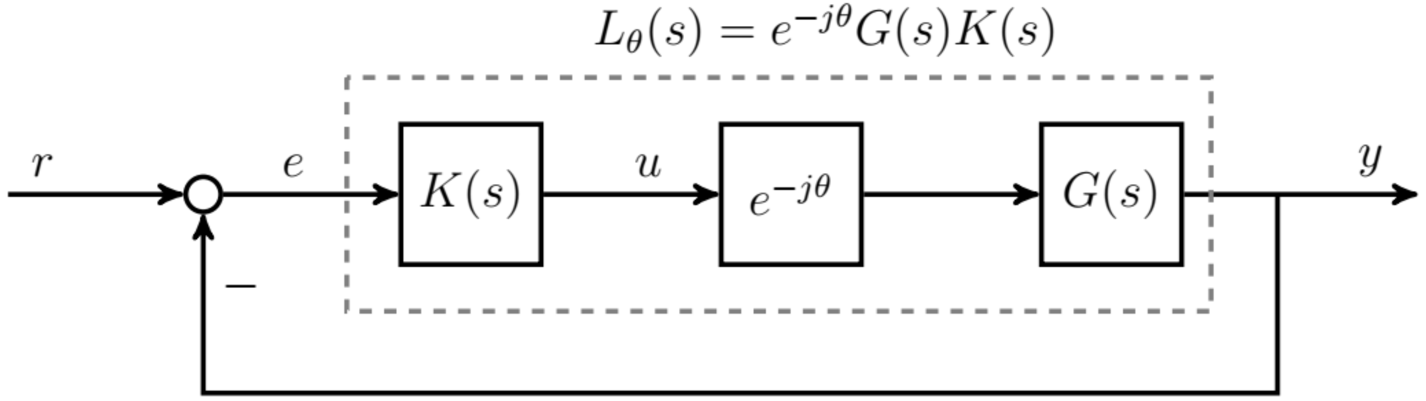
$$\Rightarrow L(s) = G(s)K(s) = \frac{2}{s^3+2s^2+3s+1}.$$

The nominal sensitivity is:

$$S(s) = \frac{1}{1+L(s)} = \frac{s^3+2s^2+3s+1}{s^3+2s^2+3s+3}.$$

The poles of S(s) are: $s_{1,2} = -0.30 \pm 1.44j$ and $s_3 = -1.39.$

These are all in the LHP so the nominal closed-loop is stable.



Example

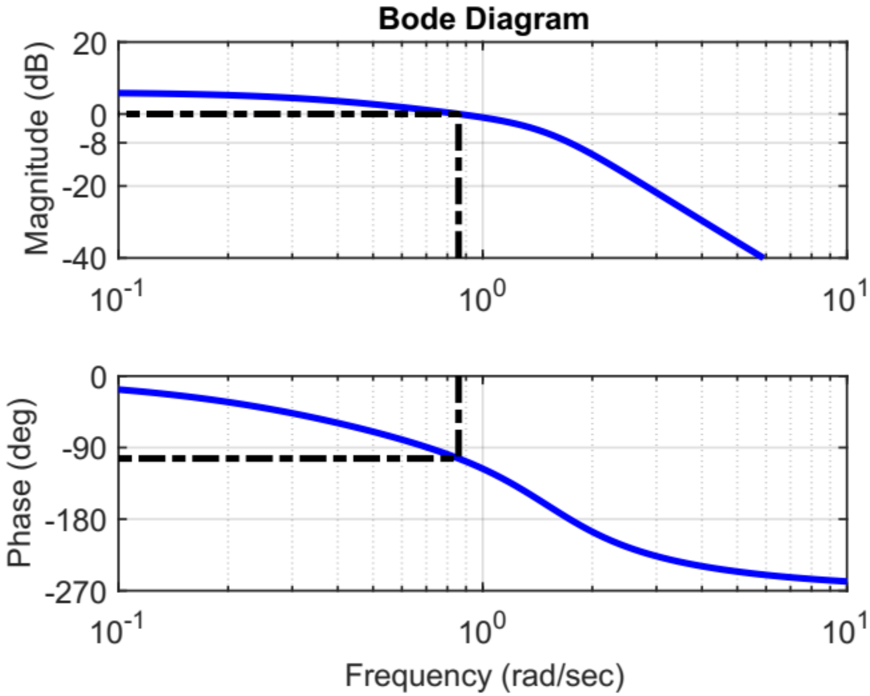
Consider the feedback system with:

$$G(s) = \frac{1}{s^3+2s^2+3s+1} \text{ and } K(s) = 2. \;\Rightarrow L(s) = G(s)K(s) = \frac{2}{s^3+2s^2+3s+1}.$$

1. There is one phase margin frequency: $\omega_0 = 0.86\frac{rad}{sec}$.
2. The loop phase is $\angle L(j\omega_0) = -103.7^o$. The critical phase is $\theta_0 = \angle L(j\omega_0) + 180^o = 76.3^o$.
3. The phase θ_0 causes a closed-loop pole at $s = \pm j\omega_0$.

The phase margin is

$\bar{\theta} = 76.3^o$.



Example

Revisit the feedback system with:

$$G(s) = \frac{1}{s^3+2s^2+3s+1} \text{ and } K(s) = 2. \;\Rightarrow L(s) = G(s)K(s) = \frac{2}{s^3+2s^2+3s+1}.$$

```
>> L = tf(2,[1 2 3 1]);
>> AM = allmargin(L);
>> theta0 = AM.PhaseMargin
76.2741
>> w0 = AM.PMFrequency
0.8586
% Verify that theta0 causes a closed-loop pole at s=+jw0
>> theta0rad = theta0*pi/180; % Convert from degs to rads
>> S0 = feedback(1, exp(-1j*theta0rad)*L);
>> pole(S0)
-0.6568 - 1.6803i
-1.3432 + 0.8216i
0.0000 + 0.8586i
```