Problem 1 (20 points). Calculate the transfer function for the following state-space model.

\[
\dot{x} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \\
y = \begin{bmatrix} 1 & 2 \end{bmatrix} x + \begin{bmatrix} 3 \end{bmatrix} u
\]

Solution.
Recall that for a system

\[
\dot{x} = Ax + Bu, \quad y = Cx + Du
\]

the transfer function is given by

\[
H(s) = C(sI - A)^{-1}B + D
\]

Here, \( A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \end{bmatrix} \) and \( D = [3] \).

Therefore,

\[
H(s) = \frac{3s^2 - 19s + 14}{s^2 - 6s + 1}
\]

Problem 2 (20 points). For the following transfer function, calculate the controllable canonical form (CCF) state-space model.

\[
G(s) = \frac{1}{(s + 2)(s^2 + 2s + 5)}
\]

Solution.
The denominator can be expanded as \( s^3 + 4s^2 + 9s + 10 \). Recall that for a transfer function:

\[
Q(s) = b_0s^n + b_1s^{n-1} + \cdots + b_{n-1}s + b_n \\
P(s) = s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n
\]

the controllable canonical realization is:

\[
A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}
\]

\[
B = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T
\]

\[
C = \begin{bmatrix} b_n - a_nb_0 & b_{n-1} - a_{n-1}b_0 & b_{n-2} - a_{n-2}b_0 & \cdots & b_1 - a_1b_0 \end{bmatrix}
\]

\[
D = [b_0]
\]

Therefore we have that,

\[
\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -9 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \\
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + [0]u
\]

Problem 3 (25 points). Prove the following:

(a) \((AB)^\top = B^\top A^\top\)

(b) \((ABC)^\top = C^\top B^\top A^\top\)
(e) \((A^T)^{-1} = (A^{-1})^T\)

(d) \((I - TAT^{-1})^{-1} = T(I - A)^{-1}T^{-1}\)

(e) For any integer \(k \geq 0\), \((TAT^{-1})^k = T A^k T^{-1}\)

Here, \((\cdot)^\top\) denotes the transpose operator and \((\cdot)^{-1}\) denotes the inverse operator. Assume matrices are invertible whenever needed.

Recall that the definition of \(A^{-1}\) is the unique matrix such that \(AA^{-1} = A^{-1}A = I\), where \(I\) is the identity matrix.

You may use previous parts to prove later parts, e.g. you may invoke Part (a) when proving Part (b).

Solution.

Note: To prove something means to definitely establish a result for all cases. Therefore, in these questions, constructing a few \(m_i \times n_i\) where \(m_i, n_i \in \{1, 2, 3, 4, \ldots\}\) example matrices and showing the identity holds is not a proof. In the following \((\cdot)^\top = (\cdot)^T\)

(a) Let \(M_{ij}\) denote the element in row \(i\) and column \(j\) of a matrix \(M\). Then note that \(M_{ji} = (M^T)_{ij}\). Therefore,

\[
(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} B^T_{ik}A^T_{kj} = (B^TA^T)_{ij}
\]

(b) By (a) we have,

\[
(ABC)^T = A(BC)^T = (BC)^TA^T = C^TB^TA^T
\]

(c) If \(B := A^{-1}\) is the inverse of \(A\) then \(AB = I = BA\). Since the identity matrix is symmetric, in particular, taking the transpose by (a) we have,

\[
B^TA^T = (AB)^T = I = (BA)^T = A^TB^T
\]

and \(B^T = (A^T)^{-1}\).

(d) Since we want to establish that

\[
(I - TAT^{-1})^{-1} = T(I - A)^{-1}T^{-1}
\]

multiply the RHS with \((I - TAT^{-1})\). We get,

\[
(I - TAT^{-1}) (T(I - A)^{-1}T^{-1}) = (T^IT^{-1} - TAT^{-1}) \left( T(I - A)^{-1}T^{-1} \right)
\]

\[
= (T(T - A^{-1})T^{-1}) \left( T(I - A)^{-1}T^{-1} \right)
\]

\[
= (T(I - A)T^{-1}) \left( T(I - A)^{-1}T^{-1} \right)
\]

\[
= T(I - A)(I - A)^{-1}T^{-1}
\]

\[
= TT^{-1}
\]

\[
= I
\]

(e) The statement trivially holds for \(k = 0\). Assume now that it holds for some \(k = n > 0\). Then,

\[
(TAT^{-1})^{k+1} = (TAT^{-1})^k (TAT^{-1}) = \left( T A^k T^{-1} \right) (TAT^{-1}) = T A^{k+1} T^{-1}
\]

Therefore, by induction it holds true for all \(k \in \mathbb{N}\).
Problem 4 (30 points). Determine whether or not the following systems are controllable. If they are controllable, put them in controllable canonical form.

(a) \[
\dot{x} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 5 & 1 & 3 & 2 \\ 6 & 1 & 3 & 4 \\ 1 & 7 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \\
y = [1 \ 0 \ 0 \ 1] x + [1] u
\]

(b) \[
\dot{x} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \\
y = [2 \ 1] x + [0] u
\]

Solution. (a) This system is not controllable ($B$ matrix is the zero matrix).

(b) This system is controllable since the controllability matrix reads as \[
\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
\] which is full rank. From (1) and (2) the transfer function is given by \[
\frac{s + 1}{s^2 - 6s + 1}.
\] Then from (3) - (6) we have that the controllable canonical form is:

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = [1 \ 1] x + [0] u
\]