ECE 486: Control Systems

Lecture 9B: PID Tuning for Second-Order Systems
Key Takeaways

This lecture describes a method to tune PID controllers using pole placement.

For second-order systems, the approach is to:

• Use PID control and
• Select the gains to place the three closed-loop poles at desired locations.
• A PI controller (without the D-term) should be used if the plant has sufficient damping.

The choice of natural frequency (time constant) is critical.
Design Approach: Pole Placement

1. Approximate the plant dynamics by a first or second-order ODE using the dominant pole approximation.

2. If the dynamics are first-order: Use a PI controller to place the two poles at a desired location.

2. If dynamics are second-order:
   - Use a PID controller to place the three poles.
   - Avoid use of derivative control if plant is well-damped. This will restrict the choice of the three poles.

A reasonable starting point is to place all poles at $s = -\omega_n$.

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2. If the dynamics are first-order: Use a PI controller to place the two poles at a desired location.

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   - Use a PID controller to place the three poles.
   - Avoid use of derivative control if plant is well-damped. This will restrict the choice of the three poles.

3. Further tuning is often required. Use root locus to tune one gain at a time.

4. Implementation:
   - D-control: Use smoothed derivative or rate feedback
   - I-control: Use anti-windup (to be discussed later)
Example plant model:

\[ \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_0 u(t) + b_0 d(t) \]

where \( a_1 = -2 \), \( a_0 = 17 \) and \( b_0 = 17 \)

Formal design requirements can be stated. Roughly a faster closed-loop response will:

- lead to better reference tracking and disturbance rejection,
- but it will also increase the actuator effort and degrade the noise rejection.

Important: Second-order ODE is typically an approximate model.

Formal tools to assess the impact of model uncertainty later.

If the closed-loop is too fast then the unmodeled dynamics will degrade performance and may even cause instability.
**Closed-Loop Model**

Dynamics of the plant:
\[ \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_0 u(t) + b_0 d(t) \]

where \( a_1 = -2 \), \( a_0 = 17 \) and \( b_0 = 17 \)

PID controller in rate feedback form:
\[ u(t) = K_p e(t) + K_i \int_0^t e(\tau) \, d\tau - K_d \dot{y}(t) \]

Sub for \( u \) into plant dynamics and collect terms.

Closed-loop dynamics are:
\[ y^{[3]}(t) + (a_1 + b_0 K_d) \ddot{y}(t) + (a_0 + b_0 K_p) \dot{y}(t) + (b_0 K_i) y(t) \]
\[ = b_0 K_p \dot{r}(t) + b_0 K_i r(t) + b_0 \dot{d}(t) \]

The closed-loop characteristic equation is:
\[ 0 = s^3 + (a_1 + b_0 K_d) s^2 + (a_0 + b_0 K_p) s + (b_0 K_i) \]
Closed-loop characteristic equation:

\[ 0 = s^3 + (a_1 + b_0K_d)s^2 + (a_0 + b_0K_p)s + (b_0K_i) \]

Pole Placement:

- Select the desired poles to satisfy for some \((\zeta, \omega_n, p)\):

\[ 0 = (s^2 + 2\zeta\omega_ns + \omega_n^2) \cdot (s + p) \]

Choose \(\zeta = 1\) and \(p = \omega_n\) as a starting point.

- The desired characteristic equation is:

\[ 0 = s^3 + (p + 2\zeta\omega_n)s^2 + (2\zeta\omega_np + \omega_n^2)s + \omega_n^2p \]

- Match coefficients to the closed-loop characteristic equation:

\[
\begin{align*}
    a_1 + b_0K_d &= p + 2\zeta\omega_n \\
    a_0 + b_0K_p &= 2\zeta\omega_np + \omega_n^2 \\
    b_0K_i &= \omega_n^2p
\end{align*}
\]

Solve these equations for the three gains.
Comparison of Two PID Controllers

**$K_1$** is designed for faster response than $K_2$.

<table>
<thead>
<tr>
<th>Design</th>
<th>$\zeta$</th>
<th>$\omega_n, \text{rad/sec}$</th>
<th>$p$</th>
<th>Poles, $s_{1,2}$ and $s_3$</th>
<th>$M_p$</th>
<th>$\tau_{\text{settle}}, \text{sec}$</th>
<th>$K_p$</th>
<th>$K_i$</th>
<th>$K_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1(s)$</td>
<td>1</td>
<td>6</td>
<td>6</td>
<td>-6, -6, -6</td>
<td>0</td>
<td>0.5</td>
<td>5.35</td>
<td>12.7</td>
<td>1.18</td>
</tr>
<tr>
<td>$K_2(s)$</td>
<td>1</td>
<td>3.5</td>
<td>3.5</td>
<td>-3.5, -3.5, -3.5</td>
<td>0</td>
<td>0.86</td>
<td>1.16</td>
<td>2.52</td>
<td>0.735</td>
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Comparison of Two PI Controllers

$K_1$ is designed for faster response than $K_2$.

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Step responses with $r(t) = 4$, $d(t) = 2$ for $t \geq 1.5$, and sensor noise for $t \geq 4$. 

![Graphs showing step responses and sensor noise](image-url)