ECE 486: Control Systems

Lecture 8B: Proportional-Derivative (PD) Control
This lecture describes proportional-derivative control. The controller sets the plant input with two terms: (i) proportional to the error and (ii) proportional to the derivative of the error.

Key properties of PD control:

1. Some plants cannot be stabilized by P or PI control. This motivates the use of PD.
2. A basic implementation of PD control will amplify noise.
3. Common implementations use a “smoothed” derivative or a direct measurement of the derivative of the output.
Rocket Attitude Control

Rockets require precise control of their heading direction (attitude) to reach their desired final destination.

$u$: Thrust angle (rad)

$y$: Heading angle (rad)
Model of Rocket Attitude Dynamics

If $|u| \ll 1$ and $|y| \ll 1$ then the dynamics are approximated by:

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_0 u(t) + b_0 d(t)$$

where: $a_1 = 0 \frac{1}{sec}$, $a_0 = -0.12 \frac{1}{sec^2}$, and $b_0 = 6.32 \frac{1}{sec^2}$

Transfer Function:

$$G(s) = \frac{6.32}{s^2 - 0.12}$$

Poles:

$$s_{1,2} = \pm 0.346 \frac{rad}{sec}$$

System is unstable

Step input with zero ICs and $u(t) = 0.05$rad
Proportional Control

Model of rocket attitude:

\[
\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_0 u(t) + b_0 d(t)
\]

where: \(a_1 = 0 \frac{1}{\text{sec}},\ a_0 = -0.12 \frac{1}{\text{sec}^2},\ \text{and}\ b_0 = 6.32 \frac{1}{\text{sec}^2}\)

Sub \(u = K_p (r - y)\) into plant model:

\[
\ddot{y}(t) + a_1 \dot{y}(t) + (a_0 + b_0 K_p) y(t) = (b_0 K_p) r(t) + b_0 d(t)
\]

The coefficient of \(\dot{y}\) is \(= 0\) and is unaffected by \(K_p\). The closed-loop will be unstable.

The rocket dynamics cannot be stabilized by P-control. Moreover it cannot be stabilized by PI-control (Routh-Hurwitz criterion can be applied to show this).
Closed-loop, proportional-derivative control for rocket:

1. User specifies the desired heading angle, $r(t)$
2. Controller computes the tracking error $e(t) = r(t) - y(t)$
3. Controller sets input thrust angle to:

$$u(t) = K_p e(t) + K_d \dot{e}(t)$$

where $K_p$ and $K_d$ are gains to be selected.
Effect of P and D Terms

P Control: \[ u(t) = K_p e(t) \]

\( K_p \) affects settling time, steady-state error, control input

PD Control: \[ u(t) = K_p e(t) + K_d \dot{e}(t) \]

Use two gains to independently modify the transient and steady-state characteristics:

P-Term: Reacts to present (current error).

D-Term: Reacts to future (derivative of error), i.e. \( \dot{e} \) indicates the direction the error is headed. Has no effect in steady-state.
Recall the second-order model for the rocket:
\[ \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_0 u(t) + b_0 d(t) \]

Substitute \( u = K_p e + K_d \dot{e} \) and combine terms:
\[
\ddot{y}(t) + (a_1 + b_0 K_d) \dot{y}(t) + (a_0 + b_0 K_p) y(t) = (b_0 K_d) \dot{r}(t) + (b_0 K_p) r(t) + b_0 d(t)
\]

This is a second-order closed-loop model from \((r,d)\) to \(y\).
Closed-Loop Response

The dynamics of the closed-loop system are:

\[ \ddot{y}(t) + (a_1 + b_0K_d) \dot{y}(t) + (a_0 + b_0K_p)y(t) = (b_0K_d) \dot{\dot{r}}(t) + (b_0K_p) \dot{r}(t) + b_0d(t) \]

\[ := 2\zeta \omega_n \]

\[ := \omega_n^2 \]

- Closed-loop is stable if and only if \( a_0 + b_0K_p > 0, a_1 + b_0K_d > 0 \).
- We can place the two closed-loop poles anywhere by proper choice of \((K_p, K_d)\). [Always true if plant is 2\textsuperscript{nd} order.]
- We are able to use the derivative term to modify the damping and stabilize the rocket attitude dynamics.
Example of PD Control

- Simulate with gains \((K_p, K_d) = (0.75, 0.47)\) and
  - \(r(t) = 0.1\text{rad}\),
  - \(d(t) = -0.01\text{rad}\) for \(t \geq 5\text{sec}\)

- Closed-loop is underdamped with:
  \[
  \omega_n = \sqrt{a_0 + b_0 K_p} = 2.14\text{sec} \quad \text{and} \quad \zeta = \frac{a_1 + b_0 K_d}{2\sqrt{a_0 + b_0 K_p}} = 0.7
  \]
Effect of Noise

• Simulate with gains \( (K_p, K_d) = (0.75, 0.47) \) and
  • \( r(t) = 0.1 \text{rad} \),
  • Sensor noise \( n(t) \) for \( t \geq 5 \text{sec} \) [Zero mean, Standard Dev=0.005]
Effect of Noise

- Simulate with gains \((K_p, K_d) = (0.75, 0.47)\) and
  - \(r(t) = 0.1 \text{rad}\),
  - Sensor noise \(n(t)\) for \(t \geq 5 \text{sec}\) [Zero mean, Standard Dev=0.005]

Derivative control can lead to large control inputs due to fast changes in the reference command or sensor noise.
Implementations for PD Control

1. Use $K_d v$ where $v$ is an approximate (smoothed) derivative:

   \[
   \dot{\hat{e}}(t) + \alpha_0 \hat{e}(t) = \alpha_0 e(t) \quad \text{with IC: } \hat{e}(0) = 0
   \]

   \[
   v(t) = \dot{\hat{e}}(t)
   \]

2. Rate-feedback implementation:

   \[
   u(t) = K_p (r(t) - y(t)) - K_d \dot{y}(t)
   \]

   This form avoids differentiating the reference. It typically uses a direct measurement of $\dot{y}$. 