ECE 486: Control Systems

Lecture 4C: Second-Order Step Response
Key Takeaways

This lecture covers the step response for second-order systems.

The step response of a *stable*, second-order system.

1. Is characterized by the natural frequency and damping ratio of the system
2. Has overshoot and oscillations if the system is underdamped.
Consider the second-order system:

\[ \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_0 u(t) \]

with \( y(0) = 0, \dot{y}(0) = 0 \)

\( u(t) = 1 \) for all \( t \geq 0 \)

The system is stable (both roots in LHP) if and only if \( a_1, a_0 > 0 \).

For stable systems, the coefficients are typically redefined:

\[ \ddot{y}(t) + 2\zeta \omega_n \dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \]

where:

- \( \omega_n \) := Natural frequency (rad/sec)
- \( \zeta \) := Damping ratio (unitless)
Overdamped/Underdamped Systems

Stable, second-order system:

\[ \ddot{y}(t) + 2\zeta \omega_n \dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \quad G(s) = \frac{b_0}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

Two poles are given by:

\[ s_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]

Three cases depending on \( \zeta^2 - 1 \):

- Overdamped, \( \zeta \geq 1 \): Roots are real and distinct
- Critically Damped, \( \zeta = 1 \): Roots are real and both at \( s_{1,2} = -\zeta \omega_n \)
- Underdamped, \( \zeta < 1 \): Roots are a complex conjugate pair.
Overdamped/Underdamped Systems

Stable, second-order system:

\[ \ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \]

Two poles are given by:

\[ s_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]

\[ G(s) = \frac{b_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

Over/Critically damped solutions (red/blue) are similar to first-order response.

Underdamped solution (yellow) has overshoot and oscillations.
If $\zeta < 1$ then the poles are:

$$s_{1,2} = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2}$$

Imaginary part $\omega_d$ is called the damped natural frequency.

Time constant is:

$$\tau = \frac{1}{\zeta \omega_n}$$

Angle $\theta$ is given by:

$$\sin(\theta) = \zeta$$

Angle decreases for smaller values of $\zeta$. 

\[\text{Diagram showing the complex plane with poles and the relationship between } \omega_d, \omega_n, \text{ and } \theta.\]
Underdamped Poles

If $\zeta < 1$ then the poles are:

$$s_{1,2} = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2}$$

Example:

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 10u(t) \quad G(s) = \frac{10}{s^2 + 2s + 10}$$

$$\omega_n^2 = 10 \Rightarrow \omega_n = \sqrt{10} \approx 3.2 \text{ rad/sec}$$

$$2\zeta \omega_n = 2 \Rightarrow \zeta = \frac{2}{2\sqrt{10}} \approx 0.32$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 3 \text{ rad/sec}$$

$$s_{1,2} = -1 \pm 3j$$
Underdamped Poles

If $\zeta < 1$ then the poles are:

$$s_{1,2} = -\zeta \omega_n \pm j\omega_n \sqrt{1 - \zeta^2}$$

Example:

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) + 10u(t) \quad G(s) = \frac{10}{s^2 + 2s + 10}$$

$$G = \text{tf}(10, [1 2 10]);$$

% Display poles, damping ratio, nat. freqs., time constants
$$\text{damp}(G)$$

<table>
<thead>
<tr>
<th>Pole</th>
<th>Damping</th>
<th>Frequency (rad/seconds)</th>
<th>Time Constant (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.00e+00 + 3.00e+00i</td>
<td>3.16e-01</td>
<td>3.16e+00</td>
<td>1.00e+00</td>
</tr>
<tr>
<td>-1.00e+00 - 3.00e+00i</td>
<td>3.16e-01</td>
<td>3.16e+00</td>
<td>1.00e+00</td>
</tr>
</tbody>
</table>
Key Features: Stable, Underdamped Step Response

\[ y(t) = \bar{y} \left( 1 - e^{-\zeta \omega_n t} \cos(\omega_d t) - e^{-\zeta \omega_n t} \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\omega_d t) \right) \text{ where } \bar{y} := \frac{b_0}{\omega_n^2} \]

- Final Value: \[ \bar{y} = G(0) \bar{u} \]
- Settling Time: \[ T_s = \frac{3}{\zeta \omega_n} \]
- Peak Overshoot: \[ M_p = e^{-\frac{\zeta}{\sqrt{1 - \zeta^2}}} \pi \text{ where } M_p = \frac{y(T_p) - \bar{y}}{\bar{y}} \]
- Peak Time: \[ T_p = \frac{\pi}{\omega_d} \]
- Rise Time: \[ T_r \approx \frac{1.8}{\omega_n} \]
Underdamped Step Response

\[ y(t) = \bar{y} \left( 1 - e^{-\zeta \omega_n t} \cos(\omega_d t) - e^{-\zeta \omega_n t} \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\omega_d t) \right) \] where \( \bar{y} := \frac{b_0}{\omega_n^2} \)
Underdamped Step Response

\[ y(t) = \bar{y} \left( 1 - e^{-\zeta \omega_n t} \cos(\omega_d t) - e^{-\zeta \omega_n t} \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\omega_d t) \right) \text{ where } \bar{y} := \frac{b_0}{\omega_n^2} \]
TD Specs in Frequency Domain

We want to visualize time-domain specs in terms of admissible pole locations for the 2nd-order system

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s + \sigma)^2 + \omega_d^2} \]

where \( \sigma = \zeta\omega_n \)

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]

Step response: \( y(t) = 1 - e^{-\sigma t} \left( \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right) \)
Rise Time in Frequency Domain

Suppose we want $t_r \leq c$ \hspace{1cm} (c is some desired given value)

$t_r \approx \frac{1.8}{\omega_n} \leq c \quad \implies \quad \omega_n \geq \frac{1.8}{c}$

Geometrically, we want poles to lie in the shaded region:

(recall that $\omega_n$ is the magnitude of the poles)
Settling Time in Frequency Domain

Suppose we want $t_s \leq c$

$$t_s \approx \frac{3}{\sigma} \leq c \quad \implies \quad \sigma \geq \frac{3}{c}$$

Want poles to be sufficiently fast (large enough magnitude of real part):

Intuition: poles far to the left $\rightarrow$ transients decay faster $\rightarrow$ smaller $t_s$