Lecture 21: introduction to state-space design.

Goal: introduce basic notions of state-space control: different state-space realizations of the same transfer function; several canonical forms of state-space systems; controllability matrix.

Reading: FPE, Chapter 7
Frequency-Domain vs. State-Space

▶ 90% of industrial controllers are designed using frequency-domain methods (PID is a popular architecture)
▶ 90% of current research in systems and control is in the state-space framework

To be able to talk to control engineers and follow progress in the field, we need to know both methods and understand the connections between them.
State-Space Methods

- the state-space approach reveals *internal system architecture* for a given transfer function
- the mathematics is different: heavy use of *linear algebra*
- this is just a short introduction; to learn this material properly, take ECE 515
A General State-Space Model

state \( x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \)  
input \( u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m \)

output \( y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \in \mathbb{R}^p \)

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

where:

\( A \) – system matrix \((n \times n)\)  
\( B \) – input matrix \((n \times m)\)  
\( C \) – output matrix \((p \times n)\)  
\( D \) – feedthrough matrix \((p \times m)\)
From State-Space to Transfer Function

Let us find the transfer function from $u$ to $y$ corresponding to the state-space model

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

- in the scalar case ($x, y, u \in \mathbb{R}$), we took the Laplace transform
- the same idea here when working with vectors: just do it component by component
From State-Space to Transfer Function

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}
\]

Recall matrix-vector multiplication:

\[
\begin{align*}
\dot{x}_i &= (Ax)_i + (Bu)_i \\
&= \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k \\
y_\ell &= (Cx)_\ell + (Du)_\ell \\
&= \sum_{j=1}^{n} c_{\ell j} x_j + \sum_{k=1}^{m} d_{\ell k} u_k
\end{align*}
\]
Now we take the Laplace transform:

\[
\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k
\]

\[
\downarrow \mathcal{L}
\]

\[
sX_i(s) - x_i(0) = \sum_{j=1}^{n} a_{ij} X_j(s) + \sum_{k=1}^{m} b_{ik} U_k(s), \quad i = 1, \ldots, n
\]

Write down in matrix-vector form:

\[
sX(s) - x(0) = AX(s) + BU(s)
\]

\[
(Is - A)X(s) = x(0) + BU(s) \quad (I \text{ is the } n \times n \text{ identity matrix})
\]

\[
X(s) = (Is - A)^{-1} x(0) + (Is - A)^{-1} BU(s)
\]
From State-Space to Transfer Function

\[
y_\ell = \sum_{j=1}^{n} c_{\ell j} x_j + \sum_{k=1}^{m} d_{\ell k} u_k
\]

\[
\downarrow \mathcal{L}
\]

\[
Y_\ell(s) = \sum_{j=1}^{n} c_{\ell j} X_j(s) + \sum_{k=1}^{m} d_{\ell k} U_k(s), \quad \ell = 1, \ldots, p
\]

Write down in matrix-vector form:

\[
Y(s) = CX(s) + DU(s)
\]

\[
= C \left[ (Is - A)^{-1} x(0) + (Is - A)^{-1} BU(s) \right] + DU(s)
\]

\[
= C(Is - A)^{-1} x(0) + \left[ C(Is - A)^{-1} B + D \right] U(s)
\]

To find the input-output t.f., set the IC to 0:

\[
Y(s) = G(s)U(s), \quad \text{where } G(s) = C(Is - A)^{-1} B + D
\]
From State-Space to Transfer Function

The transfer function from $u$ to $y$, corresponding to

$$\dot{x} = Ax + Bu$$  $$y = Cx + Du$$

is given by

$$G(s) = C(I - A)^{-1}B + D$$

Observe that $G(s)$ contains information about the state-space matrices $A, B, C, D$!!
From State-Space to Transfer Function

\[
\dot{x} = Ax + Bu \quad Y(s) = G(s)U(s) \\
y = Cx + Du = [C(I - A)^{-1}B + D]U(s)
\]

**Important!!**

- \( G(s) \) is *undefined* when the \( n \times n \) matrix \( Is - A \) is *singular* (or noninvertible), i.e., precisely when \( \det(Is - A) = 0 \)
- since \( A \) is \( n \times n \), \( \det(Is - A) \) is a *polynomial* of degree \( n \) (the *characteristic polynomial* of \( A \)):

\[
\det(Is - A) = \det \begin{pmatrix} s - a_{11} & -a_{12} & \ldots & -a_{1n} \\ -a_{21} & s - a_{22} & \ldots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \ldots & s - a_{nn} \end{pmatrix},
\]

and its roots are the *eigenvalues* of \( A \)
- \( G \) is (open-loop) *stable* if all eigenvalues of \( A \) lie in LHP.
Example: Computing $G(s)$

Consider the state-space model in **Controller Canonical Form (CCF)***::

$$
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} +
\begin{pmatrix}
0 \\
1
\end{pmatrix} u,

\begin{pmatrix}
y
\end{pmatrix} =
\begin{pmatrix}
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
$$

— this is a *single-input, single-output* (SISO) system, since $u, y \in \mathbb{R}$; the state is two-dimensional.

Let’s compute the transfer function:

$$G(s) = C(Is - A)^{-1}B$$

($D = 0$ here)

$$Is - A = \begin{pmatrix}
s & -1 \\
6 & s + 5
\end{pmatrix}$$

* We will explain this terminology later.
Example: Computing $G(s)$

$$Is - A = \begin{pmatrix} s & -1 \\ 6 & s + 5 \end{pmatrix}$$

— how do we compute $(Is - A)^{-1}$?

A useful formula for the inverse of a $2 \times 2$ matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det M \neq 0 \implies M^{-1} = \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Applying the formula, we get

$$(Is - A)^{-1} = \frac{1}{\det(Is - A)} \begin{pmatrix} s + 5 & 1 \\ -6 & s \end{pmatrix}$$

$$= \frac{1}{s^2 + 5s + 6} \begin{pmatrix} s + 5 & 1 \\ -6 & s \end{pmatrix}$$
Example: Computing $G(s)$

$$
\begin{align*}
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} &= 
\begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + 
\begin{pmatrix}
0 \\
1
\end{pmatrix} u,
\quad y = 
\begin{pmatrix}
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\end{align*}
$$

$$G(s) = C(Is - A)^{-1}B$$

$$= 
\begin{pmatrix}
1 & 1
\end{pmatrix} \frac{1}{s^2 + 5s + 6} 
\begin{pmatrix}
s + 5 & 1 \\
-6 & s
\end{pmatrix} 
\begin{pmatrix}
0 \\
1
\end{pmatrix}
$$

$$= 
\frac{1}{s^2 + 5s + 6} 
\begin{pmatrix}
1 & 1
\end{pmatrix} 
\begin{pmatrix}
1 \\
s
\end{pmatrix}
$$

$$= \frac{s + 1}{s^2 + 5s + 6}$$

▶ the above state-space model is a realization of this t.f.
▶ note how coefficients 5 and 6 appear in both $G(s)$ and $A$!!
State-Space Realizations of Transfer Functions

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} +
\begin{pmatrix}
0 \\
1
\end{pmatrix} u,
\]
\[
y = (1 & 1)
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]
\[
G(s) = \frac{s + 1}{s^2 + 5s + 6}
\]

— at least in this example, information about the state-space model \((A, B, C)\) is contained in \(G(s)\).

Is this information recoverable? — i.e., is there only one state-space realization of a given t.f.? Or are there many?

**Answer:** There are infinitely many!
State-Space Realizations of Transfer Functions

Start with

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} +
\begin{pmatrix}
0 \\
1
\end{pmatrix} u,
\]

\[
y = \begin{pmatrix}
1 \\
1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

and consider a new state-space model

\[
\dot{x} = \bar{A} x + \bar{B} u,
\]

\[
y = \bar{C} x
\]

with

\[
\bar{A} = A^T = \begin{pmatrix}
0 & -6 \\
1 & -5
\end{pmatrix},
\bar{B} = C^T = \begin{pmatrix}
1 \\
1
\end{pmatrix},
\bar{C} = B^T = \begin{pmatrix}
0 & 1
\end{pmatrix}
\]

This is a different state-space model!
State-Space Realizations of Transfer Functions

Claim: The state-space model

\[ \dot{x} = \bar{A}x + \bar{B}u, \quad y = \bar{C}x \]

with

\[ \bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{C} = B^T \]

has the same transfer function as the original model with \((A, B, C)\).

Proof:

\[
\bar{C}(Is - \bar{A})^{-1}\bar{B} = B^T \left(Is - A^T\right)^{-1} C^T \\
= B^T \left[(Is - A)^T\right]^{-1} C^T \\
= B^T \left[(Is - A)^{-1}\right]^T C^T \\
= \left[C(Is - A)^{-1}B\right]^T \\
= C(Is - A)^{-1}B
\]
State-Space Realizations of Transfer Functions

The state-space model

\[
\dot{x} = \bar{A}x + \bar{B}u, \quad y = \bar{C}x
\]

with

\[
\bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{C} = B^T
\]

has the same transfer function as the original model with \((A, B, C)\).

But the state-space model is now in the Observer Canonical Form (OCF):

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
0 & -6 \\
1 & -5
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
1 \\
1
\end{pmatrix} u, \quad y = \begin{pmatrix}
0 & 1
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]
Even More Realizations ...

Yet another realization of \( G(s) = \frac{s + 1}{s^2 + 5s + 6} \) can be extracted from the partial-fractions decomposition:

\[
G(s) = \frac{s + 1}{(s + 2)(s + 3)} = \frac{2}{s + 3} - \frac{1}{s + 2}.
\]

This is the Modal Canonical Form (MCF):

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
-3 & 0 \\
0 & -2
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
1 \\
1
\end{pmatrix} u,
\quad y = \begin{pmatrix}
2 & -1
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

Then \( C(I s - A)^{-1} B = \begin{pmatrix}
2 & -1
\end{pmatrix} \begin{pmatrix}
\frac{s + 3}{s + 2} & 0 \\
0 & \frac{1}{s + 2}
\end{pmatrix}^{-1} \begin{pmatrix}
1 \\
1
\end{pmatrix} \)

\[
= \begin{pmatrix}
2 & -1
\end{pmatrix} \begin{pmatrix}
\frac{1}{s + 3} \\
\frac{1}{s + 2}
\end{pmatrix} \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
2 & -1
\end{pmatrix} \begin{pmatrix}
\frac{1}{s + 3} \\
\frac{1}{s + 2}
\end{pmatrix} = \frac{2}{s + 3} - \frac{1}{s + 2}
\]
State-Space Realizations: Bottom Line

- a given transfer function $G(s)$ can be realized using infinitely many state-space models
- certain properties make some realizations preferable to others
- one such property is *controllability*
Controllability Matrix

Consider a single-input system \((u \in \mathbb{R})\):

\[
\dot{x} = Ax + Bu, \quad y = Cx \\
x \in \mathbb{R}^n
\]

The Controllability Matrix is defined as

\[
C(A, B) = [B \mid AB \mid A^2B \mid \ldots \mid A^{n-1}B]
\]

— recall that \(A\) is \(n \times n\) and \(B\) is \(n \times 1\), so \(C(A, B)\) is \(n \times n\);
— the controllability matrix only involves \(A\) and \(B\), not \(C\)

We say that the above system is controllable if its controllability matrix \(C(A, B)\) is invertible.

(This definition is only true for the single-input case; the multiple-input case involves the rank of \(C(A, B)\).)
Controllability Matrix

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \quad y = Cx \quad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$C(A, B) = [B \mid AB \mid A^2B \mid \ldots \mid A^{n-1}B]$$

We say that the above system is controllable if its controllability matrix $C(A, B)$ is invertible.

- As we will see later, if the system is controllable, then we may assign arbitrary closed-loop poles by state feedback of the form $u = -Kx$.
- Whether or not the system is controllable depends on its state-space realization.
Example: Computing $C(A, B)$

Let’s get back to our old friend:

$$
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
0 \\
1
\end{pmatrix} u, \\
y = \begin{pmatrix}
1 & 1
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
$$

Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2 \times 2} \implies C(A, B) \in \mathbb{R}^{2 \times 2}$

$$
C(A, B) = [B \mid AB] \\
AB = \begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix} \begin{pmatrix}
0 \\
1
\end{pmatrix} = \begin{pmatrix}
1 \\
-5
\end{pmatrix}
$$

$$
\implies C(A, B) = \begin{pmatrix}
0 & 1 \\
1 & -5
\end{pmatrix}
$$

Is this system controllable?

$$
det C = -1 \neq 0 \implies \text{system is controllable}
$$
Controller Canonical Form

A single-input state-space model

\[ \dot{x} = Ax + Bu, \quad y = Cx \]

is said to be in **Controller Canonical Form (CCF)** is the matrices \( A, B \) are of the form

\[
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
* & * & * & \ldots & * & * \\
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1 \\
\end{pmatrix}
\]

A system in CCF is always controllable!!

(The proof of this for \( n > 2 \) uses the Jordan canonical form, we will not worry about this.)
CCF with Arbitrary Zeros

In our example, we had $G(s) = \frac{s + 1}{s^2 + 5s + 6}$, with a minimum-phase zero at $z = -1$.

Let’s consider a general zero location $s = z$:

$$G(s) = \frac{s - z}{s^2 + 5s + 6}$$

This gives us a CCF realization

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-6 & -5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + 
\begin{bmatrix}
0 \\
1
\end{bmatrix} u,
\quad y = 
\begin{bmatrix}
-z & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
$$

Since $A$, $B$ are the same, $C(A, B)$ is the same $\implies$ the system is still controllable.

A system in CCF is controllable for any locations of the zeros.