Lecture 19B: lead/lag control, Part II

Goal: introduce the use of lag dynamic compensators

Reading: FPE, Chapter 5
Lead & Lag Compensators

Consider a general controller of the form

\[ K \frac{s + z}{s + p} \quad \text{— \( K, z, p > 0 \) are design parameters} \]

Depending on the relative values of \( z \) and \( p \), we call it:

- a lead compensator when \( z < p \)
- a lag compensator when \( z > p \)

Why the name “lead/lag?” — think frequency response

\[ \angle \frac{j\omega + z}{j\omega + p} = \angle (j\omega + z) - \angle (j\omega + p) = \psi - \phi \]

- if \( z < p \), then \( \psi - \phi > 0 \) (phase lead)
- if \( z > p \), then \( \psi - \phi < 0 \) (phase lag)
Lead Compensation: Bode Plot

\[ KD(s) = \frac{K \left( \frac{s}{z} + 1 \right)}{\left( \frac{s}{p} + 1 \right)} \]

- magnitude levels off at high frequencies \( \Rightarrow \) better noise suppression
- adds phase, hence the term “phase lead”
Lag Compensation: Bode Plot

\[ D(s) = \frac{s + z}{s + p} = \frac{z}{p} \frac{s + 1}{s + p} , \quad z \gg p \]

- \( j\omega + z \xrightarrow{\omega \to \infty} 1 \)
  so \( M \to 1 \) at high frequencies

- subtracts phase, hence the term “phase lag”
Lag Compensation: Bode Plot

$\frac{j\omega + z}{j\omega + p} \xrightarrow{\omega \to 0} \frac{z}{p}$

steady-state tracking error:

$$e(\infty) = \left. \frac{sR(s)}{1 + D(s)G(s)} \right|_{s=0}$$

large $z/p \implies$ better s.s. tracking

$\triangleright$ lag decreases $\omega_c \implies$ slows down time response (to compensate, adjust $K$ or add lead)

$\triangleright$ caution: lead increases PM, but adding lag can undo this

$\triangleright$ to mitigate this, choose both $z$ and $p$ very small, while maintaining desired ratio $z/p$
**Example**

\[ G(s) = \frac{1}{(s + 0.2)(s + 0.5)} \]

Bode form:

\[ G(s) = \frac{10}{(\frac{s}{0.2} + 1)(\frac{s}{0.5} + 1)} \]

**Objectives:**

- \( \text{PM} \geq 60^\circ \)
- \( e(\infty) \leq 10\% \) for constant reference (closed-loop tracking error)

**Strategy:**

- We will use lag

\[ KD(s) = K \frac{s + z}{s + p}, \quad z \gg p \]

- \( z \) and \( p \) will be chosen to get good tracking
- PM will be shaped by choosing \( K \)
- This is different from what we did for lead (used \( p \) and \( z \) to shape PM, then chose \( K \) to get desired bandwidth spec)
Review: Lead Control Using Frequency Response

General Procedure

1. Choose $K$ to get desired bandwidth spec w/o lead
2. Choose lead zero and pole to get desired PM
   ▶ in general, we should first check PM with the $K$ from 1, w/o lead, to see how much more PM we need
3. Check design and iterate until specs are met.

This is an intuitive procedure, but it’s not very precise, requires trial & error.
Step 1: Choose $K$ to Shape PM

Check Bode plot of $G(s)$ to see how much PM it already has:

- from Matlab, $\omega_c \approx 1$
- PM $\approx 40^\circ$
- we want PM $= 60^\circ$

\[
\phi = -120^\circ \quad \text{at } \omega \approx 0.573
\]
\[
M = 2.16
\]

— need to decrease $K$ to $1/2.16$

A conservative choice (to allow some slack) is $K = 1/2.5 = 0.4$, gives $\omega_c \approx 0.52$, PM $\approx 65^\circ$
Step 2: Choose $z \& p$ to Shape Tracking Error

So far: $KG(s) = \frac{0.4 \cdot 10}{\left(\frac{s}{0.2} + 1\right) \left(\frac{s}{0.5} + 1\right)}$

$$e(\infty) = \frac{1}{1 + KG(s)}\bigg|_{s=0} = \frac{1}{1 + 4} = \frac{1}{5} = 20\% \quad \text{(too high)}$$

To have $e(\infty) \leq 10\%$, need $KD(0)G(0) \geq 9$:

$$e(\infty) = \frac{1}{1 + KD(0)G(0)} \leq \frac{1}{1 + 9} = 10\%.$$ 

So, we need

$$D(0) = \frac{s + z}{s + p}\bigg|_{s=0} = \frac{z}{p} \geq \frac{9}{4} = 2.25 \quad \text{— say, } \frac{z}{p} = 2.5$$

Not to distort PM and $\omega_c$, let’s pick $z$ and $p$ an order of magnitude smaller than $\omega_c \approx 0.5$: $z = 0.05, \ p = 0.02$
Overall Design

Plant:
\[ G(s) = \frac{10}{\left(\frac{s}{0.2} + 1\right) \left(\frac{s}{0.5} + 1\right)} \]

Controller:
\[ KD(s) = 0.4 \frac{s + 0.05}{s + 0.02} \]

— the design still needs a bit of refinement ...
Lead & Lag Compensation

Let’s combine the advantages of PD/lead and PI/lag.

Back to our example: \[ G(s) = \frac{10}{\left( \frac{s}{0.2} + 1 \right) \left( \frac{s}{0.5} + 1 \right)} \]

- from Matlab, \( \omega_c \approx 1 \)
- \( \text{PM} \approx 40^\circ \)

New objectives:
- \( \omega_{BW} \geq 2 \)
- \( \text{PM} \geq 60^\circ \)
- \( e(\infty) \leq 1\% \) for const. ref.
What we got before, with lag only:

- Improved PM by adjusting $K$ to decrease $\omega_c$.
- This gave $\omega_c \approx 0.5$, whereas now we want a larger $\omega_c$ (recall: $\omega_{BW} \in [\omega_c, 2\omega_c]$, so $\omega_c = 0.5$ is too small)

So: we need to reshape the phase curve using lead.
**Lead & Lag Compensation**

Step 1. Choose $K$ to get $\omega_c \approx 2$ (before lead)

Using Matlab, can check:

at $\omega = 2$, $M \approx 0.24$ (with $K = 1$)

— need $K = \frac{1}{0.24} \approx 4.1667$

— choose $K = 4$

(gives $\omega_c$ slightly $< 2$, but still ok).
Lead & Lag Compensation

\[ K = 4 \]

\[ \text{Step 2. Decide how much phase lead is needed, and choose } z_{\text{lead}} \text{ and } p_{\text{lead}} \]

Using Matlab, can check:

\[ \text{at } \omega = 2, \quad \phi \approx -160^\circ \]

— so PM = 20°

(in fact, choosing \( K = 4 \) made things worse: it increased \( \omega_c \) and consequently decreased PM)

We need at least 40° phase lead!!

The choice of lead pole/zero must satisfy

\[ \sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4 \]
Lead & Lag Compensation

Need at least $40^\circ$ phase lead, while satisfying

\[
\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4
\]

Let’s try $z_{\text{lead}} = 1$ and $p_{\text{lead}} = 4$

\[
D(s) = \frac{s + 1}{s + \frac{4}{1}}
\]

Phase lead $= 37^\circ$ — not enough!!
Lead & Lag Compensation

Need at least $40^\circ$ phase lead, while satisfying

$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

The choice of $z_{\text{lead}} = 1, p_{\text{lead}} = 4$ gave phase lead $= 37^\circ$.

Need to space $z_{\text{lead}}$ and $p_{\text{lead}}$ farther apart:

$$\begin{cases} 
  z_{\text{lead}} = 0.8 \\
  p_{\text{lead}} = 5
\end{cases} \implies \text{phase lead } = 46^\circ$$
Lead & Lag Compensation

Step 3. Evaluate steady-state tracking and choose $z_{\text{lag}}, p_{\text{lag}}$ to satisfy specs

So far:

$$K D(s) G(s) = 4 \left( \frac{s}{0.8} + 1 \right) \cdot \frac{10}{\left( \frac{s}{0.2} + 1 \right) \left( \frac{s}{0.5} + 1 \right)}$$

$$KD(0)G(0) = 40 \implies e(\infty) = \frac{1}{1 + KD(0)G(0)} = \frac{1}{1 + 40}$$

— this is not small enough: need $1\% = \frac{1}{100} = \frac{1}{1 + 99}$

We want $D(0) \geq \frac{99}{40}$ with lag $\frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5$ will do
Lead & Lag Compensation

Need to choose lag pole/zero that are sufficiently small (not to distort the phase lead too much) and satisfy \( \frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5 \).

We can stick with our previous design:

\[
z_{\text{lag}} = 0.05, \quad p_{\text{lag}} = 0.02
\]

Overall controller:

\[
\frac{s}{4 \frac{0.8}{s} + 1} \cdot \frac{s + 0.05}{s + 0.02}
\]

\( \text{lead (with gain } K = 4 \text{ absorbed)} \)

\( \text{lag (not in Bode form)} \)

(Note: we don’t rewrite lag in Bode form, because \( z_{\text{lag}}/p_{\text{lag}} \) is not incorporated into \( K \).)
Frequency Domain Design Method: Advantages

Design based on Bode plots is good for:

- easily visualizing the concepts
- evaluating the design and seeing which way to change it
- using experimental data (frequency response of the uncontrolled system can be measured experimentally)
Frequency Domain Design Method: Disadvantages

Design based on Bode plots is not good for:

- exact closed-loop pole placement (root locus is more suitable for that)
- deciding if a given $K$ is stabilizing or not ...
  - we can only measure *how far* we are from instability (using GM or PM), if we know that we are stable
  - however, we don’t have a way of checking whether a given $K$ is stabilizing from frequency response data

The Nyquist criterion and Bode plots provide complementary benefits..