Lecture 19A: Nyquist stability criterion for varying $K$

**Goal:** learn how to detect the presence of RHP poles of the closed-loop transfer function as the gain $K$ is varied using frequency-response data

**Reading:** FPE, Chapter 6
Nyquist Stability Criterion

Goal: count the number of RHP poles (if any) of the closed-loop transfer function

\[
\frac{KG(s)}{1 + KG(s)}
\]

based on frequency-domain characteristics of the plant transfer function \(G(s)\)
Review: Nyquist Plot

Consider an arbitrary \textit{strictly proper} transfer function $H$:

$$H(s) = \frac{(s - z_1) \ldots (s - z_m)}{(s - p_1) \ldots (s - p_n)}, \quad m < n$$

\textbf{Nyquist plot:} $\text{Im } H(j\omega)$ vs. $\text{Re } H(j\omega)$ as $\omega$ varies from $-\infty$ to $\infty$
The Nyquist Stability Criterion

Nyquist Stability Criterion. Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain $K$) is stable if and only if the Nyquist plot of $G(s)$ encircles the point $-1/K$ $P$ times counterclockwise, where $P$ is the number of unstable (RHP) open-loop poles of $G(s)$. 

$$Z = N + P$$

$$Z = 0 \implies N = -P$$
Applying the Nyquist Criterion

**Workflow:**

Bode $M$ and $\phi$-plots $\rightarrow$ Nyquist plot

**Advantages of Nyquist over Routh–Hurwitz**

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- less computational, more geometric (came 55 years after Routh)
**Example**

\[
G(s) = \frac{1}{(s + 1)(s + 2)} \quad \text{(no open-loop RHP poles)}
\]

Characteristic equation:

\[
(s + 1)(s + 2) + K = 0 \quad \iff \quad s^2 + 3s + K + 2 = 0
\]

From Routh, we already know that the closed-loop system is stable for \( K > -2 \).

We will now reproduce this answer using the Nyquist criterion.
Example

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]  
(no open-loop RHP poles)

Strategy:

▶ Start with the Bode plot of \( G \)
▶ Use the Bode plot to graph \( \text{Im } G(j\omega) \) vs. \( \text{Re } G(j\omega) \) for \( 0 \leq \omega < \infty \)
▶ This gives only a portion of the entire Nyquist plot

\[ (\text{Re } G(j\omega), \text{Im } G(j\omega)), \quad -\infty < \omega < \infty \]

▶ Symmetry:

\[ G(-j\omega) = \overline{G(j\omega)} \]

— Nyquist plots are always symmetric w.r.t. the real axis!!
Example

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]

(no open-loop RHP poles)

Bode plot:

Nyquist plot:
Example: Applying the Nyquist Criterion

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]  
(no open-loop RHP poles)

Nyquist plot:

\[ \#(\odot \text{ of } -1/K) = \#(\text{RHP CL poles}) - \#(\text{RHP OL poles}) = 0 \]

\[ \implies K \in \mathbb{R} \text{ is stabilizing if and only if } \#(\odot \text{ of } -1/K) = 0 \]

- If \( K > 0 \), \( \#(\odot \text{ of } -1/K) = 0 \)
- If \( 0 < -1/K < 1/2 \), \[ \#(\odot \text{ of } -1/K) > 0 \implies \text{closed-loop stable for } K > -2 \]