Goal: learn to read off gain and phase margins of the closed-loop system from the Bode plot of the open-loop transfer function

Reading: FPE, Section 6.1
Crossover Frequency and Stability

**Definition:** The frequency at which $M = 1$ is called the *crossover frequency* and denoted by $\omega_c$.

Transition from stability to instability on the Bode plot:

For critical $K$, $\angle G(j\omega_c) = 180^\circ$
Stability from Frequency Response

Consider this unity feedback configuration:

$$G(s) Y + R K$$

Suppose that the closed-loop system, with transfer function

$$\frac{K G(s)}{1 + K G(s)}$$

is stable for a given value of $K$.

**Question:** Can we use the Bode plot to determine how far from instability we are?

Two important characteristics: gain margin (GM) and phase margin (PM).
What happens as we vary $K$?

- $\phi$ independent of $K \implies$ only the $M$-plot changes
- If we multiply $K$ by 2:
  \[
  \log(2M) = \log 2 + \log M
  \]
  - $M$-plot shifts up by $\log 2$
- If we divide $K$ by 2:
  \[
  \log\left(\frac{1}{2}M\right) = \log \frac{1}{2} + \log M
  = -\log 2 + \log M
  \]
  - $M$-plot shifts down by $\log 2$

Changing the value of $K$ moves the crossover frequency $\omega_c$!!
Gain Margin

Back to our example: \[ G(s) = \frac{1}{s(s^2 + 2s + 2)}, \quad K = 2 \text{ (stable)} \]

Gain margin (GM) is the factor by which \( K \) can be multiplied before we get \( M = 1 \) when \( \phi = 180^\circ \)

Since varying \( K \) doesn’t change \( \omega_{180^\circ} \), to find GM we need to inspect \( M \) at \( \omega = \omega_{180^\circ} \)
Gain Margin

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In this example:

At \( \omega_{180^\circ} = \sqrt{2} \)
\[ M = 0.5 (-6 \text{ dB}), \]
so \( \text{GM} = 2 \)
Phase Margin

Our example: \[ G(s) = \frac{1}{s(s^2 + 2s + 2)}, \quad K = 2 \text{ (stable)} \]

\[ \phi = -148^\circ \]

**Phase margin (PM)** is the amount by which the phase at the crossover frequency \( \omega_c \) differs from \( 180^\circ \) mod \( 360^\circ \).

To find PM, we need to inspect \( \phi \) at \( \omega = \omega_c \)

In this example:

\[ \phi = -148^\circ, \]

so \( PM = (-148^\circ) - (-180^\circ) = 32^\circ \)

(in practice, want \( PM \geq 30^\circ \))
Example

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s} \quad \zeta, \omega_n > 0 \]

Consider gain \( K = 1 \), which gives closed-loop transfer function

\[
\frac{KG(s)}{1 + KG(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s} \quad \frac{\omega_n^2}{1 + \omega_n^2} \quad \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \quad \text{— prototype 2nd-order response}
\]

Question: what is the gain margin at \( K = 1 \)?

Answer: \( GM = \infty \)
Example

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left(\frac{j\omega}{2\zeta \omega_n} + 1\right)} \]

Let’s look at the phase plot:

- starts at \(-90^\circ\) (Type 1 term with \(n = -1\))
- goes down by \(90^\circ\) (Type 2 pole)

Recall: to find GM, we first need to find \(\omega_{180^\circ}\), and here there is no such \(\omega \implies\) no GM.
Example

So, at \( K = 1 \), the gain margin of

\[
G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}
\]

is equal to \( \infty \) — what does that mean?

It means that we can keep on increasing \( K \) indefinitely without ever encountering instability.

What about phase margin?
Example: Phase Margin

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left(\frac{j\omega}{2\zeta \omega_n} + 1\right)} \]

Let’s look at the magnitude plot:

- low-frequency asymptote slope \(-1\) (Type 1 term, \(n = -1\))
- slope down by 1 past the breakpoint. \(\omega = 2\zeta \omega_n\) (Type 2 pole)

\[ \Rightarrow \text{there is a finite crossover frequency } \omega_c \]
Example 2: Magnitude Plot

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta \omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left(\frac{j\omega}{2\zeta \omega_n} + 1\right)} \]

It can be shown that, for this system,

\[ PM \bigg|_{K=1} = \tan^{-1} \left( \frac{2\zeta}{\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}} \right) \]

— for PM < 70°, a good approximation is \( PM \approx 100 \cdot \zeta \)
Phase Margin for 2nd-Order System

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left(\frac{j\omega}{2\zeta\omega_n} + 1\right)} \]

\[ \text{PM} \bigg|_{K=1} = \tan^{-1} \left( \frac{2\zeta}{\sqrt{\sqrt{4\zeta^4 + 1 - 2\zeta^2}}} \right) \approx 100 \cdot \zeta \]

Conclusions:

larger PM \iff better damping

(open-loop quantity) \iff (closed-loop characteristic)

Thus, the overshoot \( M_p = \exp \left( -\frac{\pi\zeta}{\sqrt{1-\zeta^2}} \right) \) and resonant peak \( M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1 \) are both related to PM through \( \zeta \)!!
In the next lecture, we will see the following more generally:

**Bode’s Gain-Phase Relationship**: all important characteristics of the closed-loop time response can be related to the phase margin of the open-loop transfer function!!

In fact, we will use a quantitative statement of this relationship as a design guideline.