ECE 486: Control Systems

Lecture 16A: Sensitivity Functions
Key Takeaways

This lectures considers a generic feedback system with plant $G(s)$ and controller $K(s)$.

Two important transfer functions are:

- **Sensitivity:** $S(s) = \frac{1}{1+G(s)K(s)}$

- **Complementary Sensitivity:** $T(s) = \frac{G(s)K(s)}{1+G(s)K(s)}$

The feedback system is defined to be stable if every possible input/output transfer function is internally stable.

- This holds if and only if all zeros of $1+G(s)K(s)$ are in the LHP.
- The feedback system is unstable if the $G(s)K(s)$ has a pole/zero cancellation in the CRHP.
Problem 1

Consider the feedback system below.

A) What is the transfer function from disturbance $d$ to output $y$? Express your answer in terms of $G(s)$ and $K(s)$.

B) Is the feedback system stable if $G(s) = \frac{1}{s-2}$ and $K(s) = 5$?

C) Is the feedback system stable if $G(s) = \frac{s-1}{s+2}$ and $K(s) = \frac{5}{s-1}$?
A) What is the transfer function from disturbance $d$ to output $y$? Express your answer in terms of $G(s)$ and $K(s)$.

$$Y = G(D+u) = G(D + KE) = GD + GK\cdot E$$

$$Y = GD + GK(-Y)$$

$$(1+GK)Y = GD ightarrow rac{Y}{D} = rac{G}{1+GK}$$

\[ T_d = \frac{GK}{1+GK} \]
Solution 1B

B) Is the feedback system stable if \( G(s) = \frac{1}{s-2} \) and \( K(s) = 5 \)?

Check if the system is stable if \( 1 + G(s)K(s) \) has only \( \text{LHP} \) zeros.

\[
1 + (\frac{1}{s-2})5 = \frac{(s-2) + 5}{s-2} = \frac{s+3}{s-2}
\]

The zero is at \( s = -3 \) in \( \text{LHP} \) and hence the solution is \( \text{Stable} \).

\[
T_{d-y} = \frac{G}{1+GK} \cdot \frac{1}{s-2} = \frac{1}{s+3}
\]

Diagram:

```
    r
     ↓
      e
       ↓
    K(s)      u
     ↓        ↓
      d        Unstable
     ↓        y
   G(s)
```

Analysis:

- The system is stable due to the zero at \( s = -3 \) in \( \text{LHP} \).
- The delay \( T_{d-y} \) is \( \frac{1}{s+3} \).
Solution 1C

C) Is the feedback system stable if \( G(s) = \frac{s-1}{s+2} \) and \( K(s) = \frac{5}{s-1} \)?

\[
1 + GK = 1 + \left( \frac{s-1}{s+2} \right) \left( \frac{5}{s-1} \right) = \frac{(s+2)(s-1) + 5(s-1)}{(s+2)(s-1)}
= \frac{(s+2) + 5}{(s+2)(s-1)} = \frac{(s+2)(s+1)}{(s+2)(s-1)}
\]

Zero at \( s = +1 \) in RHP \( \Rightarrow \text{Unstable} \)

\[\text{Closed Loop}\]

\[
- \quad r \quad e \quad K(s) \quad u \quad G(s) \quad y
\]
Solution 1 - Extra Space

\[ L = G \cdot K \]

\[ T_{y \rightarrow y} = \frac{L(s)}{1 + L(s)} \]

\[ T_{y \rightarrow y} = \frac{\frac{1}{s+3}}{1 + \frac{1}{s+3}} = \frac{1}{s+3} \]

\[ y + 2y = e \]

\[ y + 3y = r \]

\[ T_{y \rightarrow y_{b1}} = \frac{1}{s+3} \]

\[ y_{b1} = \frac{1}{s+3} \]

\[ L = \frac{N(s)}{D(s)} = \frac{1}{s+3} \]

\[ y + 2y = (r - y) \]

\[ (D + N) y = N R \]

\[ N \theta = \frac{N}{N + \theta} \]

\[ \theta \]

\[ y = \frac{N}{N + \theta} \]

\[ (D + N) y = N R \]

\[ T_{y \rightarrow y} = \frac{\frac{N}{N + \theta}}{1 + \frac{N}{N + \theta}} = \frac{N/0}{[N/0]} = \frac{N}{(N+\theta)} \]
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Lecture 16B: Gain Margin
This lecture discusses one safety factor called the gain margin to account for model uncertainty.

- The gain margin is the amount of allowable variation in the gain of the plant before the closed-loop becomes unstable.
- As a rule of thumb, the closed-loop should remain stable for gain variations in the range \([0.5, 2]\) (= ±6dB).

It is shown that a gain variation \(g_0 > 0\) causes a closed-loop pole at \(s = \pm j\omega_0\) if and only if \(L(j\omega_0) = -1/g_0\)

This can be used to determine gain margins from a Bode plot of the loop transfer function \(L(s)\).
Problem 2

Consider a standard closed-loop system with the loop transfer function $L(s)$ with Bode plot below. Assume the closed-loop is stable with the loop $L(s)$.

$$L(s) = \frac{-4s+72}{0.39s^2+8.02s+18}$$

A) What is the phase crossover frequency, $\omega_0$?

B) What is the gain margin, $g_0$, of the closed-loop? $g_0 = 2$

C) Is the closed-loop stable if the open-loop transfer function is $1.5L(s)$? Yes

D) Use Matlab to verify that if the loop is $g_0L(s)$ then the closed-loop has a pole at $j\omega_0$. 

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[Diagram of the closed-loop system with annotations for phase crossover frequency, gain margin, and stability verification.]
A) What is the phase crossover frequency, \( \omega_0 \)?

B) What is the gain margin, \( g_0 \), of the closed-loop?
Solution 2C and 2D

C) Is the closed-loop stable if the open-loop transfer function is $1.5L(s)$?

D) Use Matlab to verify that if the loop is $g_0L(s)$ then the closed-loop has a pole at $j\omega_0$. $L(s) = \frac{-4s+72}{0.39s^2+8.02s+18}$

\[ c) \text{ yes} \quad 1.5 < \bar{\gamma} = 2 \]
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Lecture 16C: Phase Margin
This lecture discusses another safety factor called the phase margin to account for model uncertainty and time delays.

- The phase margin is the amount of allowable variation in the phase of the plant before the closed-loop becomes unstable.
- As a rule of thumb, the closed-loop should remain stable for phase variations in the range ±45°.

It is shown that a phase variation $\theta_0 > 0$ causes a closed-loop pole at $s = j\omega_0$ if and only if $e^{-j\theta_0}L(j\omega_0) = -1$. 

This can be used to determine phase margins from a Bode plot of the loop transfer function $L(s)$. 
Consider a standard closed-loop system with the loop transfer function $L(s)$ with Bode plot below. Assume the closed-loop is stable with the loop $L(s)$.

$$L(s) = \frac{-4s + 72}{0.39s^2 + 8.02s + 18}$$

A) What is the gain crossover frequency, $\omega_0$?
B) What is the phase margin, $\theta_0$, of the closed-loop?
C) Use Matlab to verify that if the loop is $e^{-j\theta_0}L(s)$ then the closed-loop has a pole at $j\omega_0$.
D) What is the delay margin, $\tau_0$, of the closed-loop?
E) Use Matlab to verify (via simulation) that that closed-loop is unstable with a delay of $\tau_0$. 
Solution 3A and 3B

A) What is the gain crossover frequency, $\omega_0$?

B) What is the phase margin, $\theta_0$, of the closed-loop?

$\omega_0 = 10 \text{ rad/sec}$

$\theta_0 = 45^\circ$

$e^{-j\theta_0} L(j\omega) = -1$

$1 + e^{j\theta_0} L(j\omega) = 0$
Solution 3C

C) Use Matlab to verify that if the loop is $e^{-j\theta_0}L(s)$ then the closed-loop has a pole at $j\omega_0$. 

![Magnitude and Phase Plot]

- Magnitude (dB) vs Frequency (rad/sec)
- Phase (deg) vs Frequency (rad/sec)