ECE 486: Control Systems

Lecture 16B: Gain Margin

Key Takeaways

This lecture discusses one safety factor called the gain margin to account for model uncertainty.

- The gain margin is the amount of allowable variation in the gain of the plant before the closed-loop becomes unstable.
- As a rule of thumb, the closed-loop should remain stable for gain variations in the range [0.5, 2] (= ±6dB).

It is shown that a gain variation $g_0>0$ causes a closed-loop pole at $s = \pm j\omega_0$ if and only if $L(j\omega_0) = -1/g_0$

This can be used to determine gain margins from a Bode plot of the loop transfer function $L(s)$.

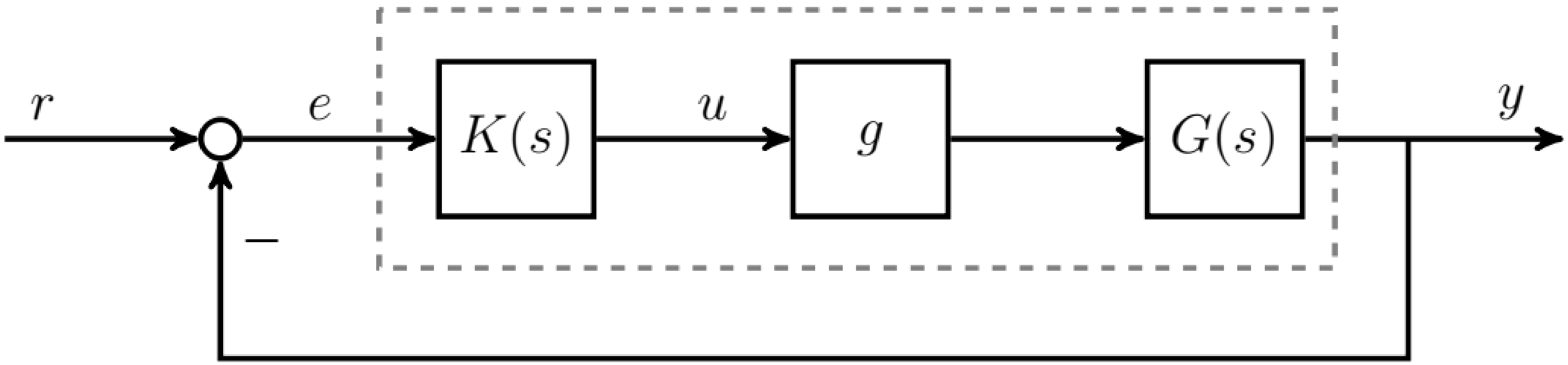
Gain Margin

Control engineers have developed different types of “safety factors" to account for the mismatch between the design model and the dynamics of the real system.

One type of safety factor called the gain margin:

- Design model G(s) might differ from real dynamics by a gain *g>0*.
- Assume the closed-loop is stable with the nominal model (*g=1*).
- The closed-loop may become unstable as *g* is varied away from *g=1*.
- The gain margin measures the variation before instability occurs.

$$L_g(s) = gG(s)K(s)$$



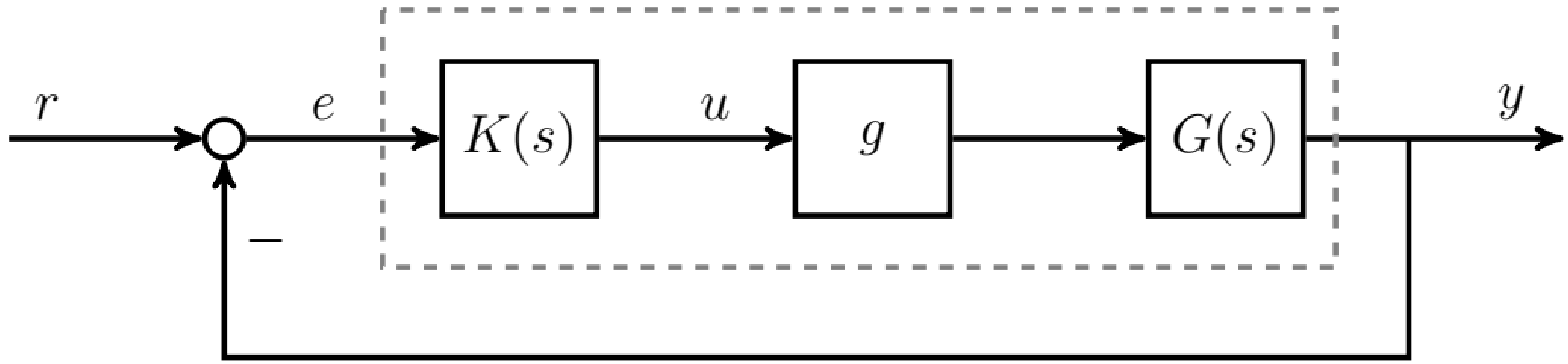
Gain Margin

Definition: The **gain margin** consists of an upper limit $\bar{g} > 1$ and a lower limit $\underline{g} < 1$ such that:

1. the closed-loop is stable for all positive gain variations g in the range $\underline{g} < g < \bar{g}$, and
2. the closed-loop is unstable for gain variations $g = \bar{g}$ (if $\bar{g} < \infty$) and $g = \underline{g}$ (if $\underline{g} = 0$).

As a rule of thumb, the closed-loop should remain stable for gain variations in the range [0.5, 2] (= ±6dB).

$$L_g(s) = gG(s)K(s)$$



Example

Consider the feedback system with:

$$G(s) = \frac{1}{s^3+2s^2+3s+1} \text{ and } K(s) = 2.$$

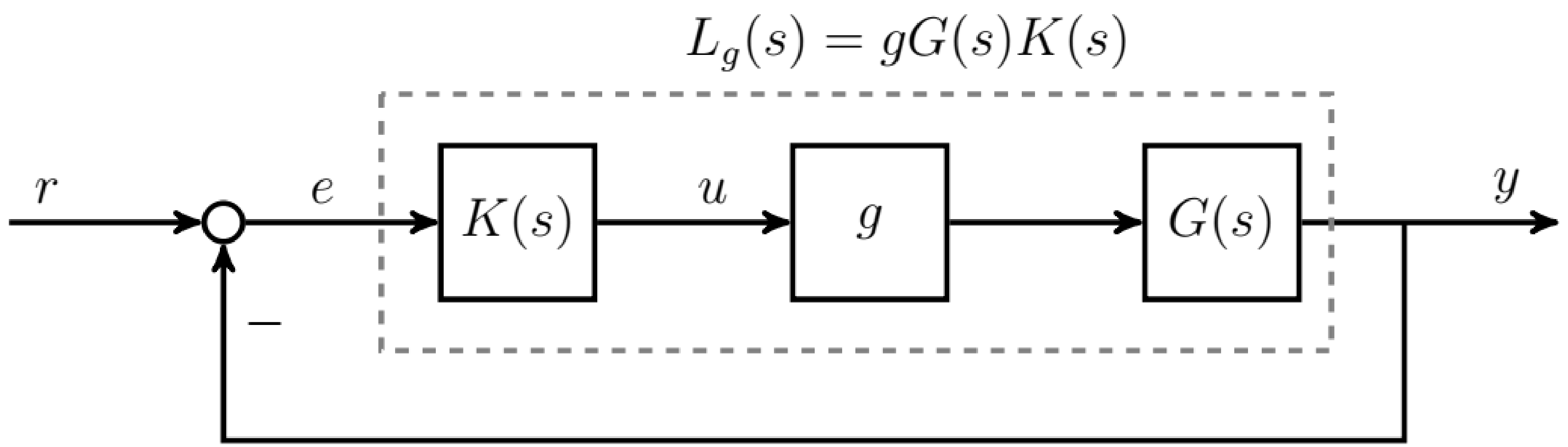
$$\Rightarrow L(s) = G(s)K(s) = \frac{2}{s^3+2s^2+3s+1}.$$

The nominal sensitivity is:

$$S(s) = \frac{1}{1+L(s)} = \frac{s^3+2s^2+3s+1}{s^3+2s^2+3s+3}.$$

The poles of S(s) are: $s_{1,2} = -0.30 \pm 1.44j$ and $s_3 = -1.39.$

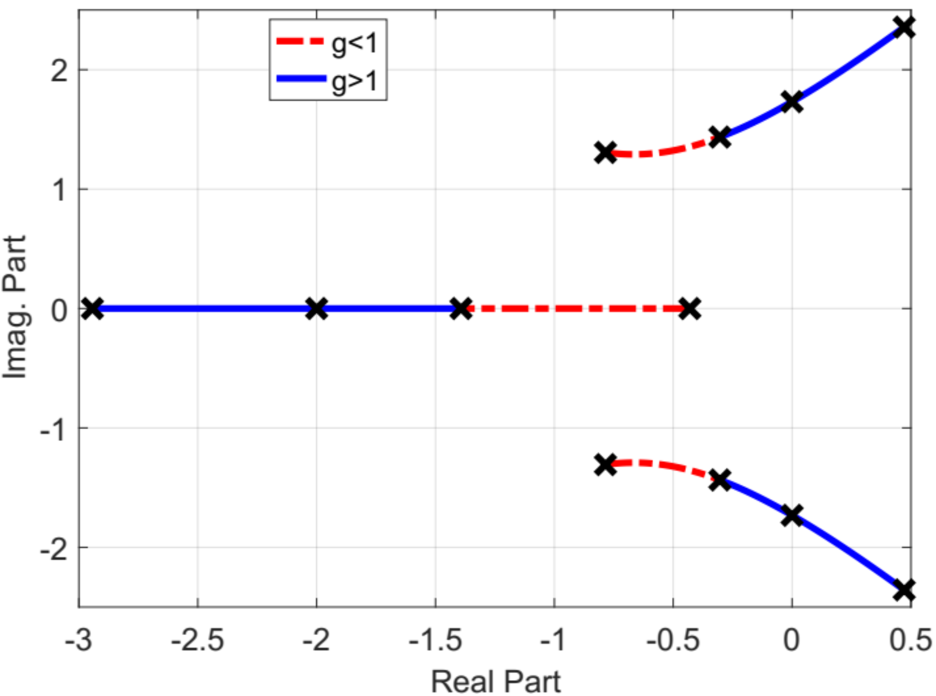
These are all in the LHP so the nominal closed-loop is stable.



Example

Poles move as gain g varies away from 1:

- Closed-loop is stable for all 0<g<1 ⇒ $\underline{g} = 0$.
- Poles cross into RHP as gain is increased. Boundary occurs for $\bar{g} = 2.5 \approx 8.5dB$ with poles at $s = \pm 1.73j$.



$G(s) = \frac{1}{s^3+2s^2+3s+1}$ and $K(s) = 2$.

g	Pole 1	Pole 2	Pole 3
0.0	-0.78 + 1.31j	-0.78 - 1.31j	-0.43
1.0	-0.30 + 1.44j	-0.30 - 1.44j	-1.39
2.5	+1.73j	-1.73j	-2.00
8.0	0.47 + 2.36j	0.47 - 2.36j	-2.94

Critical Gains

The closed-loop poles move continuously in the complex plane as the gain g is varied away from 1.

Critical gains are values of g for which the closed-loop poles are on the imaginary axis. These gains mark the transition between stable (LHP) and unstable (RHP).

- A critical gain g_0 causes a closed-loop pole at $s = \pm j\omega_0$.
- A critical gain g_0 causes $1 + g_0 L(j\omega_0) = 0$.

A critical gain g_0 >0 cause a closed-loop pole at $s = \pm j\omega_0$ if and only if $L(j\omega_0) = -\frac{1}{g_0}$.

Connection to Bode Plots

A critical gain g_0 >0 cause a closed-loop pole at $s = \pm j\omega_0$ if and only if $L(j\omega_0) = -\frac{1}{g_0}$. Identify on a Bode plot by:

1. Find frequencies where $\angle L(\omega_0) = -180^o$. These are called *phase crossing or gain margin frequencies.*
2. The critical gain is $\mathrm{g}_0 = \frac{1}{|L(j\omega_0)|}$.
3. This critical gain causes the closed-loop poles at $s = \pm j\omega_0$.

The upper gain margin $\bar{g}$ corresponds to the smallest critical gain > 1 (if any exist). The lower gain margin $\underline{g}$ corresponds to the largest critical gain < 1 (if any exist).

The Matlab function `allmargin` computes all critical gains and corresponding phase crossing frequencies.

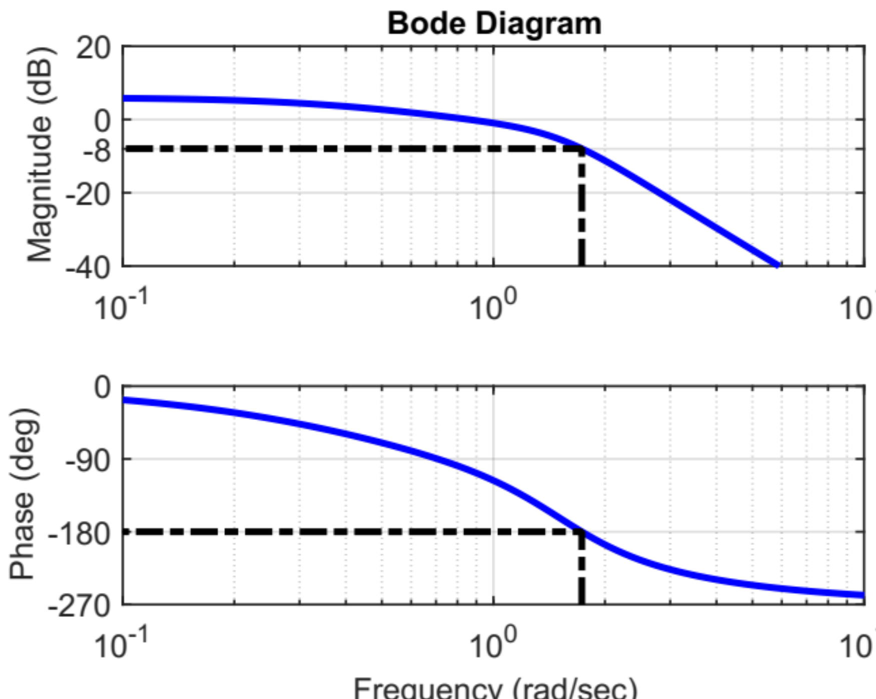
Example

Revisit the feedback system with:

$$G(s) = \frac{1}{s^3+2s^2+3s+1} \text{ and } K(s) = 2. \Rightarrow L(s) = G(s)K(s) = \frac{2}{s^3+2s^2+3s+1}.$$

1. There is one gain margin frequency: $\omega_0 = 1.73 \frac{rad}{sec}$.
2. The loop gain is $|L(j\omega_0)| = 0.4 = -8dB$. The critical gain is $g_0 = \frac{1}{|L(j\omega_0)|}$=2.5=+8$dB$.
3. The gain g_0 causes a closed-loop pole at $s = \pm j\omega_0$.

The gain margins are $\underline{g} = 0$ and $\bar{g} =$ 2.5=+8dB.



Example

Revisit the feedback system with:

$$G(s) = \frac{1}{s^3+2s^2+3s+1} \text{ and } K(s) = 2. \ \Rightarrow L(s) = G(s)K(s) = \frac{2}{s^3+2s^2+3s+1}.$$

```
>> L = tf(2,[1 2 3 1]);
>> AM = allmargin(L);
>> g0 = AM.GainMargin
2.5000
>> w0 = AM.GMFrequency
1.7321

% Verify that g0 causes a closed-loop pole at s=+/-jw0
>> S0 = feedback(1,g0*L);
>> pole(S0)
-2.0000 + 0.0000i
0.0000 + 1.7321i
0.0000 - 1.7321i
```