ECE 486: Control Systems

- Lecture 15: Bode plots for three types of transfer functions and general LTI systems

Goal: learn to analyze and sketch magnitude and phase plots of transfer functions written in Bode form (arbitrary products of three types of factors).

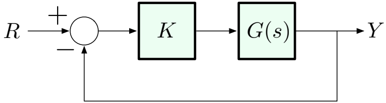
Reading: FPE, Section 6.1

Review: Scale Convention for Bode Plots

	magnitude	phase
horizontal scale	log	log
vertical scale	log	linear

Advantage of the scale convention: we will learn to do Bode plots by starting from simple factors and then building up to general transfer functions by considering products of these simple factors.

Preview: Bode's Gain-Phase Relationship



Assuming that $G(s)$ is *minimum-phase* (i.e., has no RHP zeros), we derived the following for the Bode plot of $KG(s)$:

	low freq.	real zero/pole	complex zero/pole
mag. slope	n	up/down by 1	up/down by 2
phase	$n \times 90°$	up/down by $90°$	up/down by $180°$

We can state this succinctly as follows:

Gain-Phase Relationship. Far enough from break-points,

$$\text{Phase} \approx \text{Magnitude Slope} \times 90°$$

Bode Form of the Transfer Function

Bode form of $KG(s)$ is a factored form with the constant term in each factor equal to 1, i.e., lump all DC gains into one number in the front.

Example:

$$KG(s) = K \frac{s+3}{s(s^2+2s+4)}$$

$$\text{rewrite as} \quad \left. \frac{3K\left(\frac{s}{3}+1\right)}{4s\left(\left(\frac{s}{2}\right)^2+\frac{s}{2}+1\right)} \right|_{s=j\omega}$$

$$= \underbrace{\frac{3K}{4}}_{=K_0} \frac{\frac{j\omega}{3}+1}{j\omega\left(\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right)}$$

Three Types of Factors

Transfer functions in Bode form will have three types of factors:

1. $K_0(j\omega)^n$, where n is a positive or negative integer
2. $(j\omega\tau + 1)^{\pm 1}$
3. $\left[\left(\dfrac{j\omega}{\omega_n}\right)^2 + 2\zeta\dfrac{j\omega}{\omega_n} + 1\right]^{\pm 1}$

In our example above,

$$
\begin{aligned}
KG(j\omega) &= \frac{3K}{4}\frac{\frac{j\omega}{3}+1}{j\omega\left[\left(\frac{j\omega}{2}\right)^2 + \frac{j\omega}{2} + 1\right]} \\
&= \underbrace{\frac{3K}{4}(j\omega)^{-1}}_{\text{Type 1}} \cdot \underbrace{\left(\frac{j\omega}{3}+1\right)}_{\text{Type 2}} \cdot \underbrace{\left[\left(\frac{j\omega}{2}\right)^2 + \frac{j\omega}{2} + 1\right]^{-1}}_{\text{Type 3}}
\end{aligned}
$$

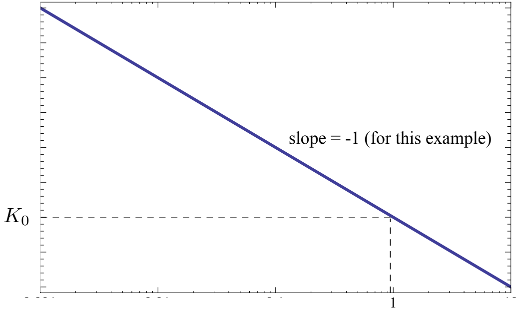
Now let's discuss Bode plots for factors of each type.

Type 1: $K_0(j\omega)^n$

Magnitude: $\log M = \log |K_0(j\omega)^n| = \log |K_0| + n \log \omega$

— as a function of $\log \omega$, this is a *line* of slope n passing through the value $\log |K_0|$ at $\omega = 1$

In our example, we had $K_0(j\omega)^{-1}$:

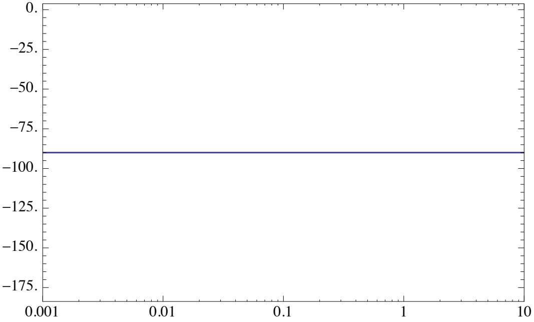


— this is called a low-frequency asymptote (will see why later)

Type 1: $K_0(j\omega)^n$

Phase: $\angle K_0(j\omega)^n = \angle(j\omega)^n = n\angle j\omega = n \cdot 90^\circ$
— this is a constant, independent of ω.

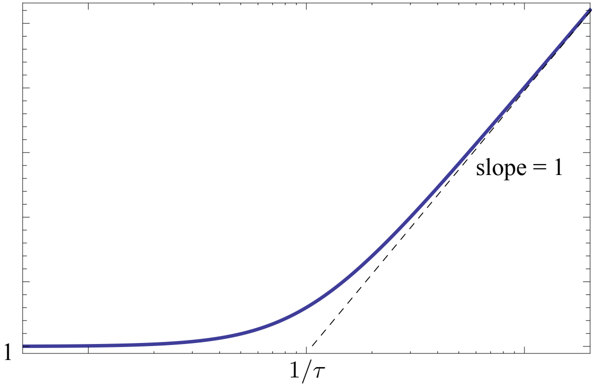
In our example, we had $K_0(j\omega)^{-1}$:



— here, the phase is -90° for all ω.

Type 2: $j\omega\tau + 1$

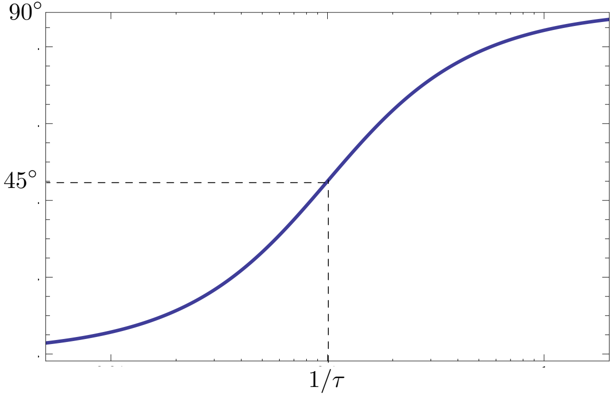
Magnitude plot:



For a stable real zero, the magnitude slope "steps up by 1" at the break-point.

Type 2: $j\omega\tau + 1$

Phase plot:



For a stable real zero, the phase "steps up by 90°" as we go past the break-point.

Type 2: $(j\omega\tau + 1)^{-1}$

This is a stable real pole.

Magnitude:

$$\log \left| \frac{1}{j\omega\tau + 1} \right| = -\log |j\omega\tau + 1|$$

Phase:

$$\angle \frac{1}{j\omega\tau + 1} = -\angle(j\omega\tau + 1)$$

So the magnitude and phase plots for a stable real pole are the reflections of the corresponding plots for the stable real zero w.r.t. the horizontal axis:

- step down by 1 in magnitude slope
- step down by 90° in phase

Example: Type 1 and Type 2 Factors

$$KG(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}$$

Convert to Bode form:

$$KG(j\omega) = \frac{2000 \cdot 0.5 \cdot \left(\frac{j\omega}{0.5}+1\right)}{10 \cdot 50 \cdot j\omega \left(\frac{j\omega}{10}+1\right)\left(\frac{j\omega}{50}+1\right)}$$

$$= \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5}+1\right) \cdot \frac{1}{\left(\frac{j\omega}{10}+1\right)\left(\frac{j\omega}{50}+1\right)}$$

Example 1: Magnitude

Transfer function in Bode form:

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1 \right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1 \right) \left(\frac{j\omega}{50} + 1 \right)}$$

Type 1 term:

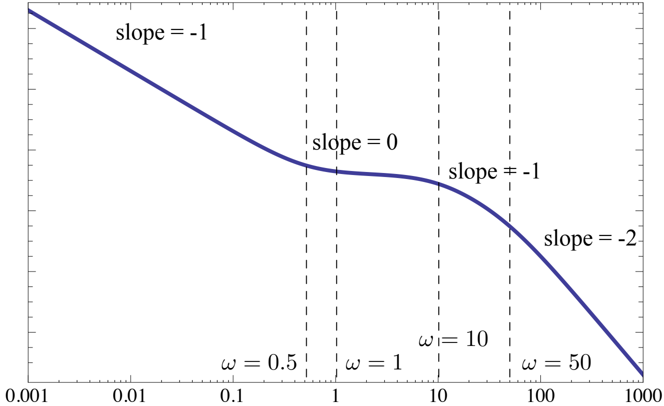
- $K_0 = 2, n = -1$ — it contributes a line of slope -1 passing through the point $(\omega = 1, M = 2)$.
- This is a low-frequency asymptote: for small ω, it gives very large values of M, while other terms for small ω are close to $M = 1$ (since $\log 1 = 0$).

Now we mark the break-points, from Type 2 terms:

- $\omega = 0.5$ stable zero $\Rightarrow$ slope steps up by 1
- $\omega = 10$ stable pole $\Rightarrow$ slope steps down by 1
- $\omega = 50$ stable pole $\Rightarrow$ slope steps down by 1

Example 1: Magnitude Plot

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$



Example 1: Phase

Transfer function in Bode form:

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$

Type 1 term:

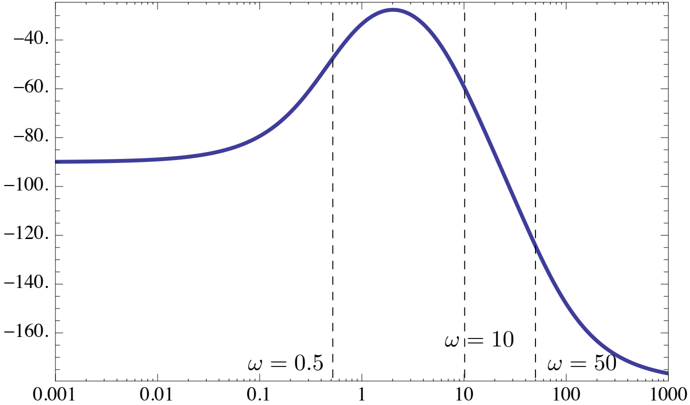
- $n = -1$ — phase starts at -90°

Type 2 terms:

- $\omega = 0.5$ stable zero $\Rightarrow$ phase up by 90° (by 45° at $\omega = 0.5$)
- $\omega = 10$ stable pole $\Rightarrow$ phase down by 90° (by 45° at $\omega = 10$)
- $\omega = 50$ stable pole $\Rightarrow$ phase down by 90° (by 45° at $\omega = 50$)

Example 1: Phase Plot

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$



Type 3: $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{-1}$

This is a stable complex pole.

Magnitude:

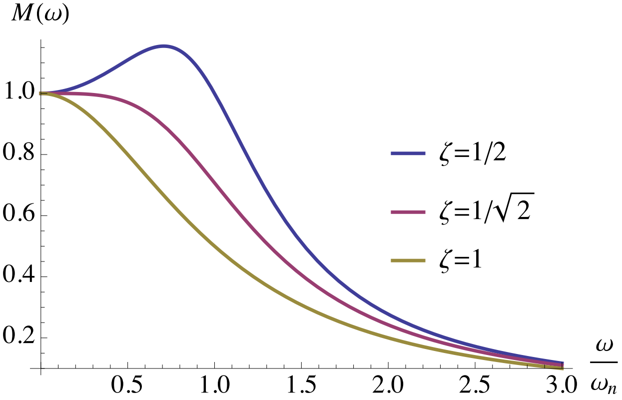
$$\log M = \log\left|\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1}\right| = -\log\left|\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right|$$

Phase:

$$\phi = \angle\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1} = -\angle\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]$$

Type 3: Magnitude, Complex Pole Case

How does the magnitude plot look? Depends on the value of ζ:



The magnitude hits its peak value (for $\zeta < 1/\sqrt{2} \approx 0.707$) at

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} < \omega_n$$

Type 3: Magnitude

For small enough ζ (below $1/\sqrt{2}$), the magnitude of

$$\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1}$$

has a resonant peak at the resonant frequency

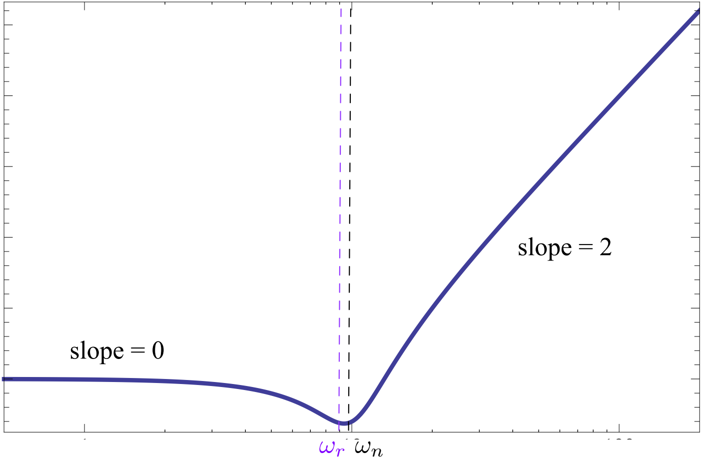
$$\omega_r = \omega_n\sqrt{1-2\zeta^2}.$$

Likewise, the magnitude of

$$\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$$

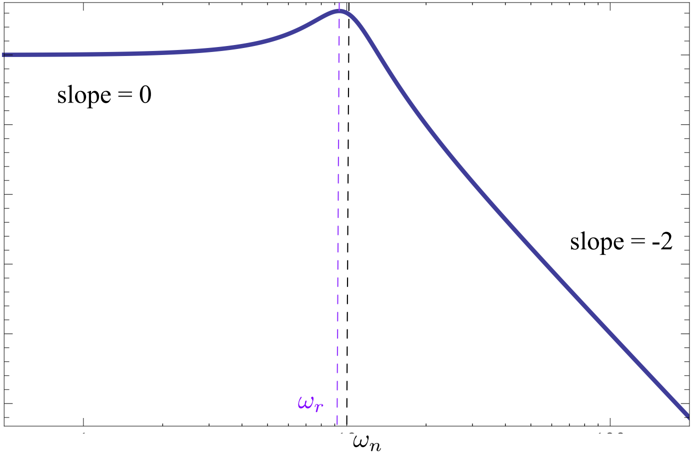
has a resonant dip at ω_r.

Type 3 Zero: Magnitude



For a stable real zero, the magnitude slope “steps up by 2” at the break-point.

Type 3 Pole: Magnitude



For a stable real pole, the magnitude slope "steps down by 2" at the break-point.

Type 3: $\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$, Phase

Take a look at the real and imaginary parts of $\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$

$$
\begin{aligned}
&(R(\omega), I(\omega)) \\
&= \left(1 - \left(\frac{\omega}{\omega_n}\right)^2, 2\zeta\frac{\omega}{\omega_n}\right)
\end{aligned}
$$

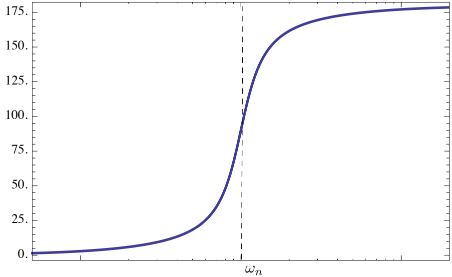
Phase:

- for $\omega \ll \omega_n$, $\phi \approx 0^\circ$ (real and positive)
- for $\omega = \omega_n$, $\phi = 90^\circ$ (Re $= 0$, Im > 0)
- for $\omega \gg \omega_n$, $\phi \approx 180^\circ$ (Re $\sim -\omega^2$, Im $\sim \omega$)

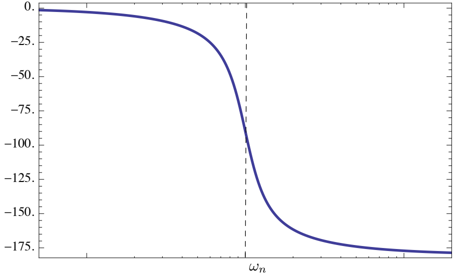
For a stable complex zero, the phase steps up by 180° as we go through the breakpoint; as $\zeta \to 0$, the transition through the break-point gets sharper, almost step-like.

For a pole, the phase is multiplied by -1.

Type 3: Phase



(stable complex zero — phase steps up by 180°)



(stable complex pole — phase steps down by 180°)

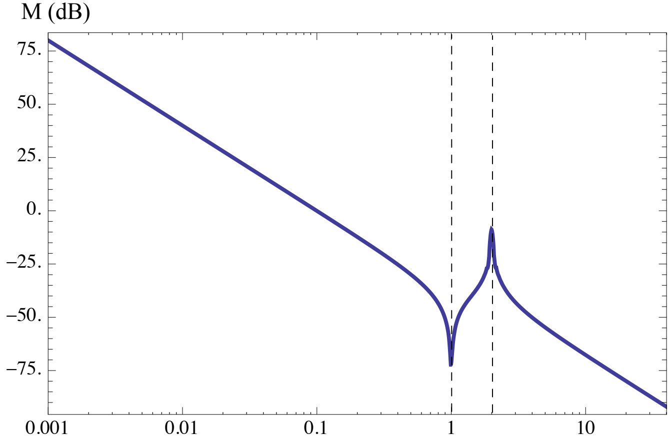
Example 2

$$KG(s) = \frac{0.01\left(s^2 + 0.01s + 1\right)}{s^2\left(\frac{s^2}{4} + 0.02\frac{s}{2} + 1\right)} \qquad \text{— already in Bode form}$$

What can we tell about magnitude?

- low-frequency term $\frac{0.01}{(j\omega)^2}$ with $K_0 = 0.01$, $n = -2$ — asymptote has slope $= -2$, passes through $(\omega = 1, M = 0.01)$
- complex zero with break-point at $\omega_n = 1$ and $\zeta = 0.005$ — slope up by 2; large resonant dip
- complex pole with break-point at $\omega_n = 2$ and $\zeta = 0.01$ — slope down by 2; large resonant peak

Example 2: Magnitude Plot



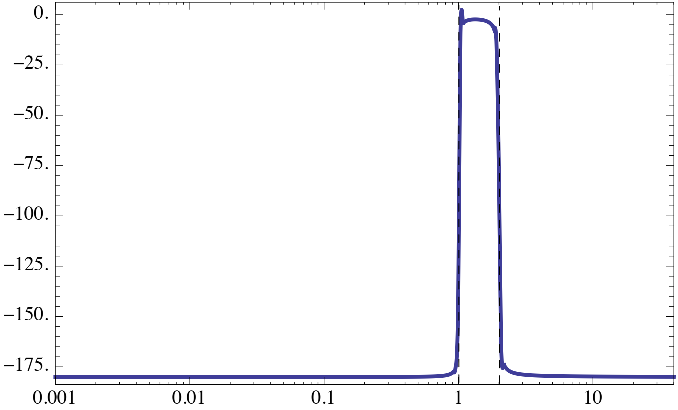
Example 2

$$KG(s) = \frac{0.01\left(s^2 + 0.01s + 1\right)}{s^2\left(\dfrac{s^2}{4} + 0.02\dfrac{s}{2} + 1\right)}$$ — already in Bode form

What can we tell about phase?

- low-frequency term $\dfrac{0.01}{(j\omega)^2}$ with $K_0 = 0.01$, $n = -2$
  — phase starts at $n \times 90^\circ = -180^\circ$
- complex zero with break-point at $\omega_n = 1$ — phase up by 180°
- complex pole with break-point at $\omega_n = 2$ — phase down by 180°
- since ζ is small for both pole and zero, the transitions are very sharp

Example 2: Phase Plot



Unstable Zeros/Poles?

So far, we've only looked at transfer functions with stable poles and zeros (except perhaps at the origin). What about RHP?

Example: consider two transfer functions,

$$G_1(s) = \frac{s+1}{s+5} \qquad \text{and} \qquad G_2(s) = \frac{s-1}{s+5}$$

Note:

- G_1 has stable poles and zeros; G_2 has a RHP zero.
- Magnitude plots of G_1 and G_2 are the same —

$$|G_1(j\omega)| = \left|\frac{j\omega+1}{j\omega+5}\right| = \sqrt{\frac{\omega^2+1}{\omega^2+5}}$$

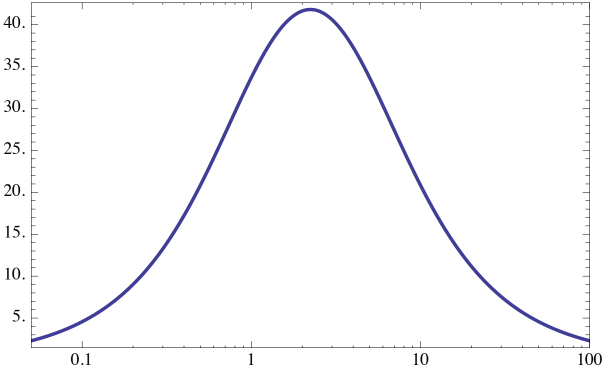
$$|G_2(j\omega)| = \left|\frac{j\omega-1}{j\omega+5}\right| = \sqrt{\frac{\omega^2+1}{\omega^2+5}}$$

- All the difference is in the phase plots!

Phase Plot for G_1

$$G_1(j\omega) = \frac{j\omega + 1}{j\omega + 5} = \frac{1}{5}\frac{j\omega + 1}{\frac{j\omega}{5} + 1}$$

- Low-frequency term: $\frac{1}{5}(j\omega)^0$— $n = 0$, so phase starts at $0°$
- Break-points at $\omega_n = 1$ (phase goes up by $90°$) and at $\omega_n = 5$ (phase goes down by $90°$)



Phase Plot for G_2

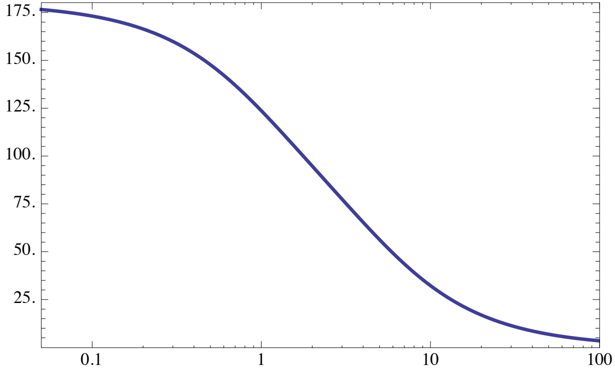
$$G_2(j\omega) = \frac{j\omega - 1}{j\omega + 5} = \frac{1}{5}\frac{j\omega - 1}{\frac{j\omega}{5} + 1}$$

New type of behavior —

- $\omega \approx 0$: $\phi \approx 180^\circ$ (real and negative)
- $\omega \gg 1$: $\phi \approx 90^\circ$ (Re $= -1$, Im $= \omega \gg 1$)
- $\omega \approx 1$: $\phi \approx 135^\circ$

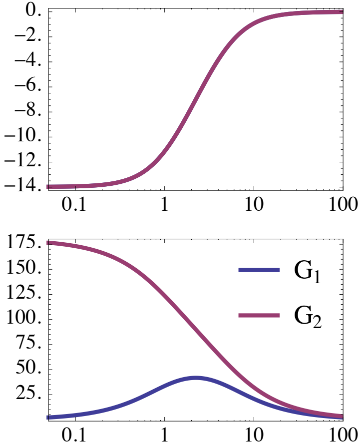
For a RHP zero, the phase starts out at 180° and goes down by 90° through the break-point (135° at break-point).

Phase Plot for G_2



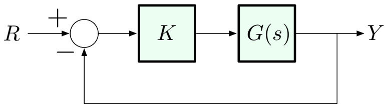
For a RHP zero, the phase plot is similar to what we had for a LHP pole: goes down by 90° ... However, it starts at 180°, and not at 0°.

Minimum-Phase and Nonminimum-Phase Zeros



Among all transfer functions with the same magnitude plot, the one with only LHP zeros has the minimal net phase change as ω goes from 0 to ∞ — hence the term *minimum-phase* for LHP zeros.

Bode's Gain-Phase Relationship



Assuming that $G(s)$ is *minimum-phase* (i.e., has no RHP zeros), we derived the following for the Bode plot of $KG(s)$:

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We can state this succinctly as follows:

Gain-Phase Relationship. Far enough from break-points,

$$\text{Phase} \approx \text{Magnitude Slope} \times 90°$$