ECE 486: Control Systems

Lecture 14A: Bode Plots for Second-Order Systems
Key Takeaways

This lecture focuses on Bode plots for second order systems.

- Second-order differentiator $G(s) = s^2$: Phase is $+180^\circ$ and magnitude has slope $+40\text{dB/decade}$.

- Second-order integrator $G(s) = \frac{1}{s^2}$: Phase is $-180^\circ$ and magnitude has slope $-40\text{dB/decade}$.

- Second-order underdamped $G(s) = \frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}$:
  - Magnitude is (approximately) flat up to the corner frequency $\omega_n$ and rolls off at $-40\text{dB/dec}$ at high frequencies.
  - Phase plot transitions by $\pm180^\circ$ depending on the signs of the coefficients.
  - If damping is low ($\zeta \ll 1$) then the plot has a resonant peak of $|G(j\omega_n)| \approx \frac{1}{2\zeta}$. The boxed equation should be: $|G(j\omega_n)| = \frac{|G(0)|}{2\zeta}$.
Sketch approximate, straight line Bode plots for the following systems:

A) \( G(s) = \frac{2}{s^2 + 2\zeta s + 4} \) with \( \zeta = 0.7 \).

\( \omega_n = 2 \) and
\( G(\omega_n) = \frac{2}{(2)^2 + 2(0.7) + 4} = \frac{2}{8} = \frac{1}{4} \).

\( \|G(\omega_n)\| = \frac{2}{\sqrt{2}} = \sqrt{2} \).

B) \( G(s) = \frac{2}{s^2 + 2\zeta s + 4} \) with \( \zeta = 0.01 \).

\( \omega_n = 2 \) and
\( G(\omega_n) = \frac{2}{(2)^2 + 2(0.01) + 4} = \frac{2}{18} = \frac{1}{9} \).

\( \|G(\omega_n)\| = \frac{2}{\sqrt{18}} = \frac{1}{3\sqrt{2}} \).

C) \( G(s) = s^2 + 2\zeta s + 4 \) with \( \zeta = 0.01 \).
Solution 1A

A) \( G(s) = \frac{2}{s^2 + 2\zeta s + 4} \) with \( \zeta = 0.7 \).

\[ G(0) = \frac{2}{4} = \frac{1}{2} \]

\[ G(j\omega) \rightarrow \frac{2}{j\omega} \sim -\frac{2}{\omega^2} \]

\[ G(j\omega) = \frac{2}{(j\omega)^2 - 12(j\omega) + 4} = \frac{1}{2j\omega} - \frac{j}{2\pi} + \frac{34}{30} \]

\[ G(j\omega) \sim \frac{1}{\omega} \]

\[ G(j\omega) = \frac{1}{2\pi} \]

\[ G(j\omega) = -90^\circ \]
Solution 1B

B) \( G(s) = \frac{2}{s^2 + 2\zeta s + 4} \) with \( \zeta = 0.01 \).

Note (Added after lecture)

Standard Form is \( \frac{2}{s^2 + 2\omega_n s + \omega_n^2} \)

The transfer function above is not in standard form

\[ G(s) = \frac{2}{s^2 + 2(0.01)s + 0.01} = \frac{2}{s^2 + 2(0.005)s + (2)^2} \]

In other words, it actually has \( \omega_n = 2 \), \( \zeta = 0.005 \).

I will review this at the start of next lecture.
Problem 1A/B

3A/B) \( G(s) = \frac{2}{s^2 + 2\zeta s + 4} \) with \( \zeta = 0.7 \) and \( \zeta = 0.01 \).

\[
G(s) = \frac{2}{s^2 + 2(0.01)s + 4} = \frac{2}{s^2 + 2(0.005)0.2s + 4}
\]

\( \omega_n^2 = 4 \) w/\( \omega_n = 2 \)

\( G(0) = \frac{2}{4} = 0.5 \approx -6 \text{dB} \)

\( \omega_n = 2 \)

\[ |G(j\omega_n)| \approx \left| \frac{G(s)}{2\pi} \right| \]

\[ \frac{1}{\omega_n} = 100 = +40 \text{dB} \]

\[ 2 \]

\[ 20 \]

\[ -46 \text{dB} \]
C) \[ G(s) = \frac{s^2}{\xi^2} + 2\zeta s + \frac{4}{\xi^2} \text{ with } \zeta = 0.01. \]

\[ G(0) = 4^2 = +8 \text{dB} \quad \angle G(0) = 0^\circ \]

\[ \omega \to \infty \quad G(j\omega) = (j\omega)^2 = -\omega^2 \quad \angle G(j\omega) = -180^\circ \]

\[ \omega_n = 2 \quad G(j2) = (j2)^2 + 2\frac{4}{2} + 4 = 4j \cdot j \]

\[ \frac{4G(j\omega_n)}{2} = +90^\circ \quad |G(j2)| = 4 \quad \begin{array}{c}
A\text{dB} \\
0.04 \\
-28 \text{ dB}
\end{array} \]
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Lecture 14B: Frequency Content of Signals
Key Takeaways

Bode plots can be used to gain intuition for how the system will respond to “low frequency” and “high frequency” signals.

The intuition follows from the following facts:

• The steady-state sinusoidal response for a stable system can be computed using the transfer function.

• By linear superposition, if the input is a sum of sinusoids then the steady state response is given by summing the responses due to each input sinusoid.

• General signals can be expressed as a sum of sinusoids using the Fourier Series. Signals can be roughly classified as low or high frequency based on the Fourier Series coefficients.
Problem 2

Consider the following first-order system: \[ G(s) = \frac{2}{s+2} \]

Is each input signal below roughly “high frequency” (fast) or “low frequency” slow for this system? Roughly sketch the output.

[Hint: You don’t need to work formally with the Fourier Series. Just consider the time constant of the system.]

\( \text{Pole at } s = -2 \quad \tau \approx \frac{1}{2} \text{ sec} \)

Input A "Fast"

Input B "Slow"
% Matlab code
% System
G = tf(2,[1 2]);

% Input A
N=1e3; t1=linspace(0,0.30,N);
u1=t1/0.1; u1(t1>0.10 & t1<=0.20) = 1; idx=find(t1>0.20); u1(idx) = 1-(t1(idx)-0.20)/0.10;
[y1,t1] = lsim(G,u1,t1);
figure(1)
plot(t1,u1,'b',t1,y1,'r--')

% Input B
N=1e3; t2=linspace(0,30,N);
u2=t2/10; u2(t2>10 & t2<=20) = 1; idx=find(t2>20); u2(idx) = 1-(t2(idx)-20)/10;
[y2,t2] = lsim(G,u2,t2);
figure(2)
plot(t2,u2,'b',t2,y2,'r--')
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Lecture 14C: Introduction to Bode Plots for Higher-Order Systems
Consider a system whose transfer function is \( G(s) = G_1(s)G_2(s) \).

- The Bode phase plot of \( G(s) \) is the sum of the phase plots of \( G_1(s) \) and \( G_2(s) \).
- The Bode magnitude plot of \( G(s) \) (in dB) is the sum of the magnitude plots of \( G_1(s) \) and \( G_2(s) \).

This can be used to draw Bode plots for higher order systems.
Problem 3

Sketch approximate, straight-line Bode plots for the following systems:

A) \[ G(s) = \frac{3s+2}{s+6} = (3s+2)(\frac{1}{s+6}) \]

B) \[ G(s) = \frac{6}{s^2+4s+3} = \frac{6}{(s+3)(s+1)} = \left(\frac{6}{5+3}\right) \left(\frac{1}{s+1}\right) \]
A) \[ G(s) = \frac{3s+2}{s+6} = \left( \frac{3s+12}{G_1} \right) \cdot \left( \frac{1}{576} \right) = \frac{26dB}{G_2} \]

\[ G_1(j\omega) = 3\omega \]

\[ G_2(j\omega) = \frac{1}{6} = -15.6dB \]

\[ \omega \rightarrow 6 \quad G_2(j\omega) \rightarrow \frac{1}{3} \]

\[ j\omega \rightarrow \frac{1}{\omega} \]
B) \[ G(s) = \frac{6}{s^2 + 4s + 3} = \left( \frac{6}{s+1} \right) \left( \frac{1}{s+1} \right) \]
\[
\frac{3s+2}{s+6} = \frac{(3s+2)}{(s+6)} \\
= \frac{\frac{3s+2}{s}}{\frac{1}{s}} \quad G_1 \quad G_2
\]

\[G_1(s) \rightarrow 3s \]
\[G_1(s) = 3s + 2 \]
\[G_1(6) = 2 \quad \circ \quad G_2(\frac{1}{s}) \]
\[G_2(s) = \frac{1}{s+6} \quad G_2(s) = \frac{1}{6} = -15.16 \text{ dB} \]
\[ g(x) = \left( \frac{6}{x+3} \right) \left( \frac{1}{x+1} \right) \]

\[ G_1 \quad G_2 \]

\[ 20 \log_{10} 10 = +20 \text{ dB} \]
\[ 20 \log_{10} 121 = +6 \text{ dB} \]
\[ 20 \log_{10} \frac{1}{2} = -6 \text{ dB} \]
\[ 20 \log_{10} \frac{1}{10} = -20 \text{ dB} \]
Problem 4

Consider a feedback system with the plant and controller:

\[ G(s) = \frac{2}{s+2} \quad K(s) = \frac{s+0.5}{s} \]

A) What type of controller has the transfer function \( K(s) \)?

B) Sketch the Bode magnitude plot of \( G(s) \), \( K(s) \), and \( G(s)K(s) \).
Consider a feedback system with the plant and controller:

\[ G(s) = \frac{2}{s+2} \quad K(s) = \frac{s+0.5}{s} = (s+1/2) \cdot \frac{1}{2} \]

A) What type of controller has the transfer function \( K(s) \)?
Consider a feedback system with the plant and controller:

\[ G(s) = \frac{2}{s+2} \quad \text{and} \quad K(s) = \frac{s+0.5}{s} = (s+0.5)\left(\frac{1}{s}\right) \]

B) Sketch the Bode magnitude plot of \( G(s) \), \( K(s) \), and \( G(s)K(s) \).