ECE 486: Control Systems

Lecture 13A: Steady-State Sinusoidal Response

Key Takeaways

The transfer function *G(s)* is used to express the solution of a stable linear system forced by a sinusoidal input.

If the input is $u(t) = \sin(\omega t)$ then the response satisfies:

$$y(t) \to |G(j\omega)|\sin(\,\omega t + \angle G(j\omega)\,) \text{ as } t \to \infty$$

Assuming G is stable

The output converges to a sinusoid at the same frequency as the input but with amplitude scaled by $|G(j\omega)|$ and phase is shifted by $\angle G(j\omega)$.

Problem 1

Consider the following first-order system and sinusoidal input:

$$-2\dot{y}(t) - y(t) = 3u(t) \qquad u(t) = 5\sin(4t + 0.1)$$

A) What is the magnitude and phase of *G(jω)*?

B) Is the steady-state response bounded? If yes, what is it?

Consider the following first-order system and sinusoidal input:

$$-2\dot{y}(t) + y(t) = 3u(t) \qquad u(t) = 5\sin(4t + 0.1)$$

C) What is the magnitude and phase of *G(jω)*?

D) Is the steady-state response bounded? If yes, what is it?

Solution 1A and 1B

Consider the following first-order system and sinusoidal input:

$$-2\dot{y}(t) - y(t) = 3u(t) \qquad u(t) = 5\sin(\underset{\omega}{\underline{4}}t + 0.1)$$

A) What is the magnitude and phase of *G(j$\underline{\omega}$)*?

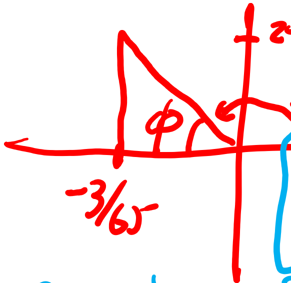
B) Is the steady-state response bounded? If yes, what is it?

Ⓐ $G(s) = \frac{3}{-2s-1} \longrightarrow -2s-1=0 \rightarrow s = -1/2$ Stable)

$G(4j) = \frac{3}{-2(4j)-1} = \frac{3}{-8j-1} \cdot \left(\frac{8j-1}{-8j-1}\right) = \frac{-3+24j}{65}$

$|G(4j)| = \sqrt{(3/65)^2 + (24/65)^2} = 0.37$

$\rightarrow \measuredangle G(4j) = 1.70$



$\measuredangle G = \pi - \phi$

$\boxed{\tan\phi = -24/3}$

Correction: Should have been $\tan\phi = 24/3$

$\rightarrow \phi = 1.45$

$\measuredangle G = \pi - \phi = 1.70$

Ⓑ yes, $y(t) \rightarrow 5|G(4j)|\sin(4t + 0.1 + \measuredangle G(4j))$

$= 1.85\sin(4t + 1.80)$

Solution 1C and 1D

Consider the following first-order system and sinusoidal input:

$$-2\dot{y}(t)+y(t)=3u(t) \qquad u(t)=5\sin(\underbrace{4}_{\omega}t+0.1)$$

C) What is the magnitude and phase of *G(jω)*?

D) Is the steady-state response bounded? If yes, what is it?

Ⓒ $G(s)=\dfrac{3}{-2s+1} \rightarrow G(4j)=\dfrac{3}{-2(4j)+1}=0.046+0.369j$

$|G(j4)|=0.37, \quad \measuredangle G(j4)=1.44$

Ⓓ $-2s+1=0 \rightarrow s=+1/2$ Unstable

No, response is not bounded

Solution 1-Extra Space

Problem 2

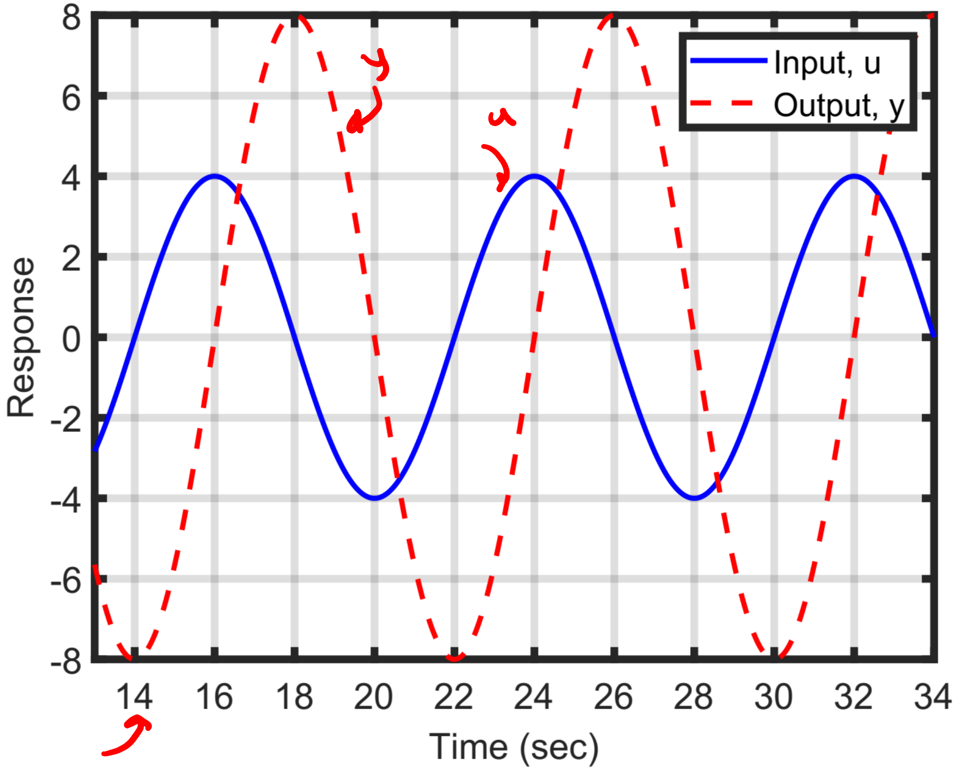
The figure shows the output y(t) generated by a linear system G(s) with input u(t) = $\underline{A_0}$ cos($\underline{\omega_0}$t).

A) What are the values of A_0 and ω_0 for the input signal u(t)?

B) What is the magnitude $|G(j\omega_0)|$?

C) What is the phase $\angle G(j\omega_0)$ in degrees?

$$\xrightarrow{u} \boxed{G} \rightarrow y$$



Solution 2A

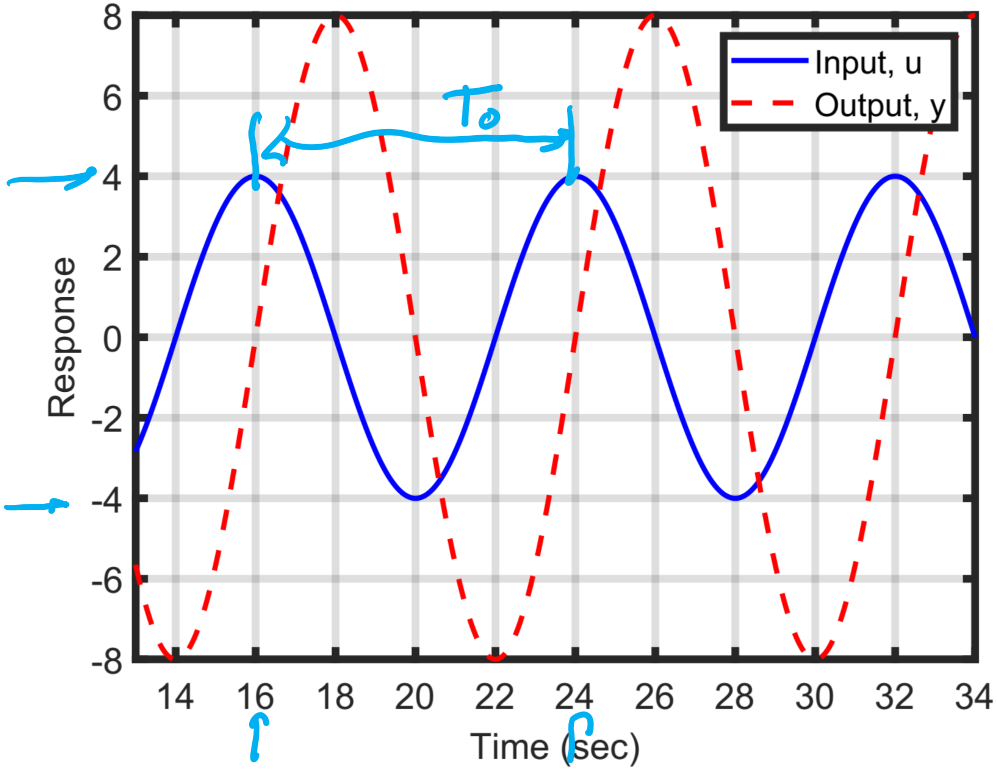
A) What are the values of A_0 and ω_0 for the input signal u(t)?

$A_0 = 4$

$u(t) = A_0 \cos \omega_0 t$

$T_0 = 8 sec$

$\omega_0 = \frac{2\pi}{T} = \pi/4$ rads/sec



Solution 2B and 2C

B) What is the magnitude $|G(j\omega_0)|$? $\omega_0 = \pi/4$ rad/sec

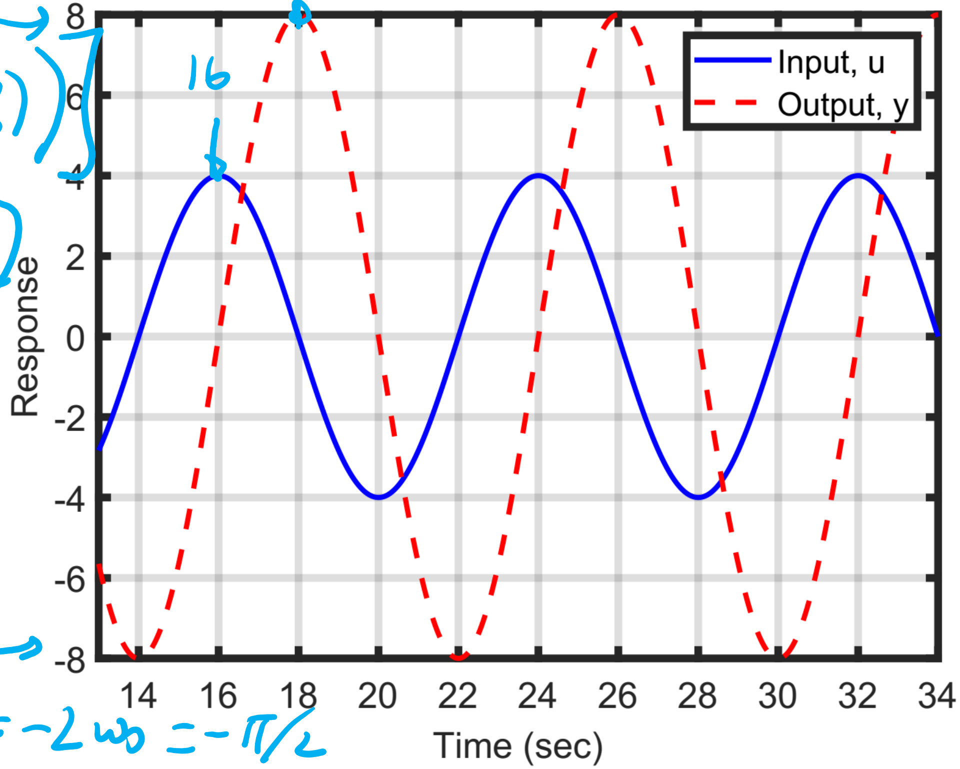
C) What is the phase $\angle G(j\omega_0)$ in degrees? $\measuredangle G(j\omega_0) = -\frac{\pi}{2}$ rad $\times \frac{180}{\pi} = -90^\circ$

$$y(t) \rightarrow \underbrace{|G(j\omega_0)| \overset{4}{A}}_{8} \cos(\underbrace{\omega_0 t + \measuredangle G(j\omega_0)}_{\cos(\omega_0(t + \measuredangle G/\omega_0))})$$

18

"Lagging" $(\measuredangle G < 0)$

$$|G(j\omega_0)| = 2$$



$$y(t_y) \rightarrow |G(j\omega_0)| \left[\underbrace{A \cos\left(\omega_0\left(t + \frac{\measuredangle G}{\omega_0}\right)\right)}\right]$$

$$= |G(j\omega_0)|\, u(\underbrace{t_y + \measuredangle G/\omega_0})$$

$$t_u = t_y + \measuredangle G/\omega_0$$

Peak Time for u

Peak Time for y

$$16 = 18 + \measuredangle G/\omega_0 \rightarrow \measuredangle G = -2\omega_0 = -\pi/2$$

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Lecture 13B: Bode Plots

Key Takeaways

A Bode plot for an LTI system *G(s)* consists of two subplots:

- Magnitude (Gain) vs. frequency and
- Phase vs. frequency.

Such plots are useful to understand the steady-state response of the system *G(s)* to sinusoids of different frequencies.

Problem 3

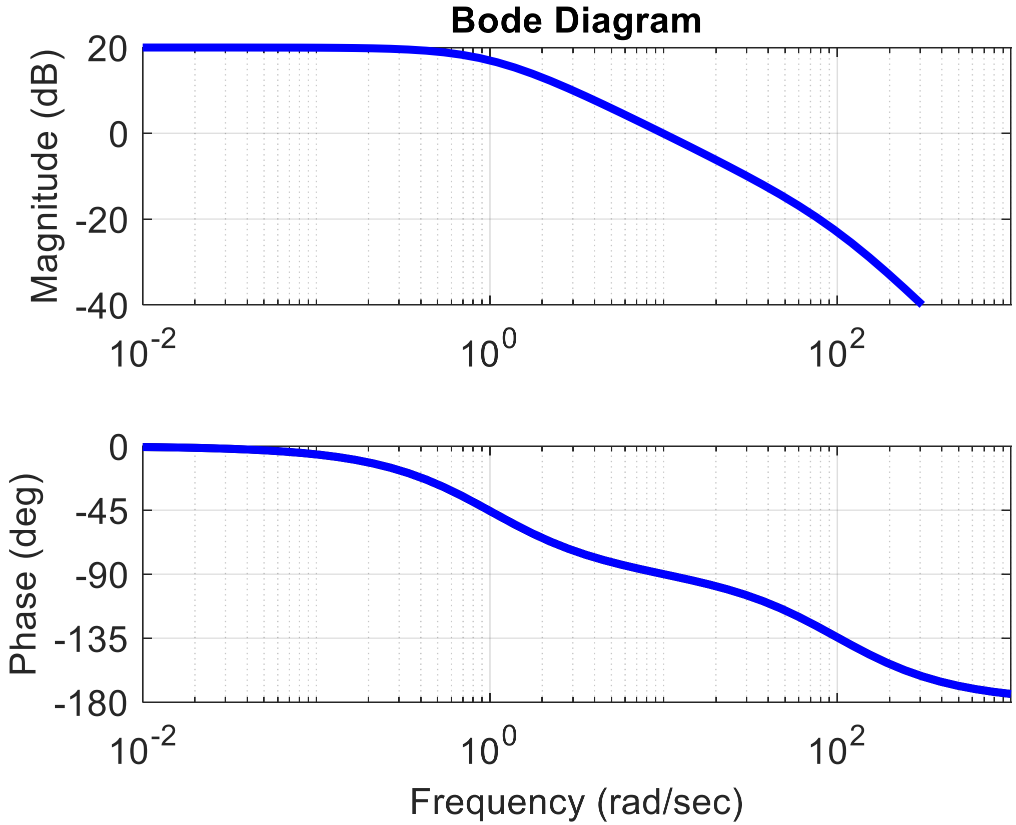
A linear system G(s) with input u and output y has the Bode plot shown below.

A) What is $|G(10j)|$ in dB and actual units?

B) What is $\angle G(10j)$ in degs and radians?

C) What is the output response *y(t)* in steady-state for the input *u(t) = 2 cos(10t)*?

D) What is the steady-state value of y(t) if the input is a unit step u(t) = 1 for all t ≥ 0?



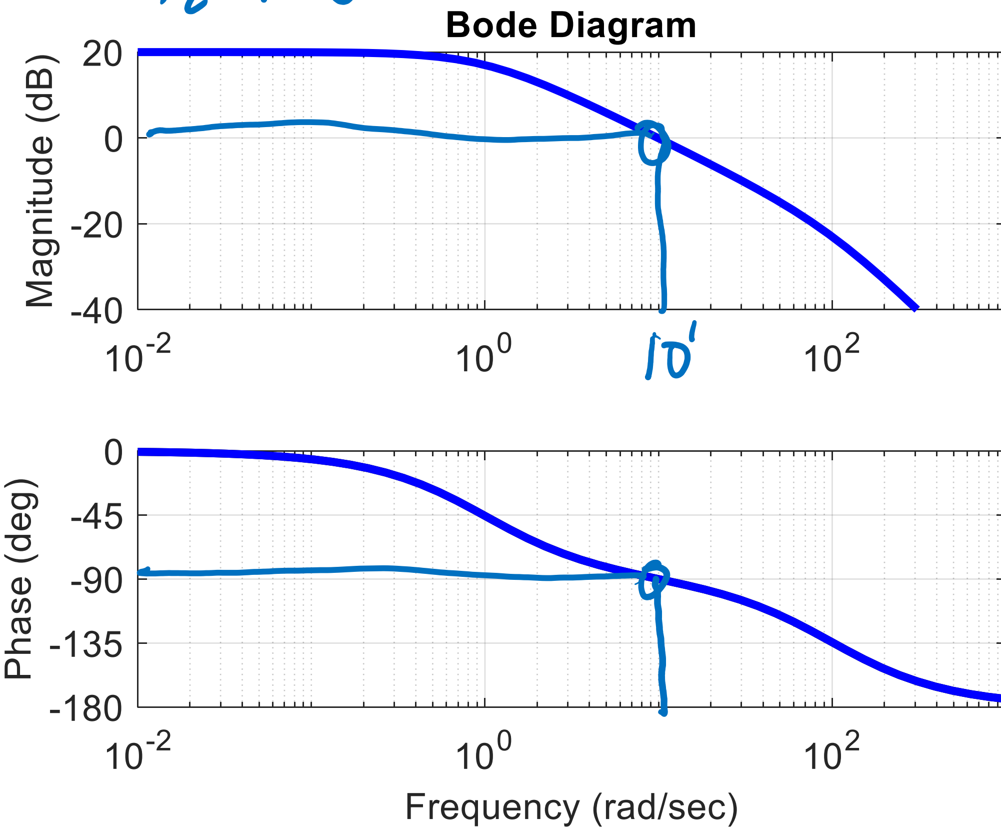
Solution 3A and 3B

A) What is $|G(10j)|$ in dB and actual units?

B) What is $\angle G(10j)$ in degs and radians?

$|G(j10)| = 0\,dB = 1$

$\angle G(j10) = -90\,deg \times \pi/180 = -\pi/2$ rads

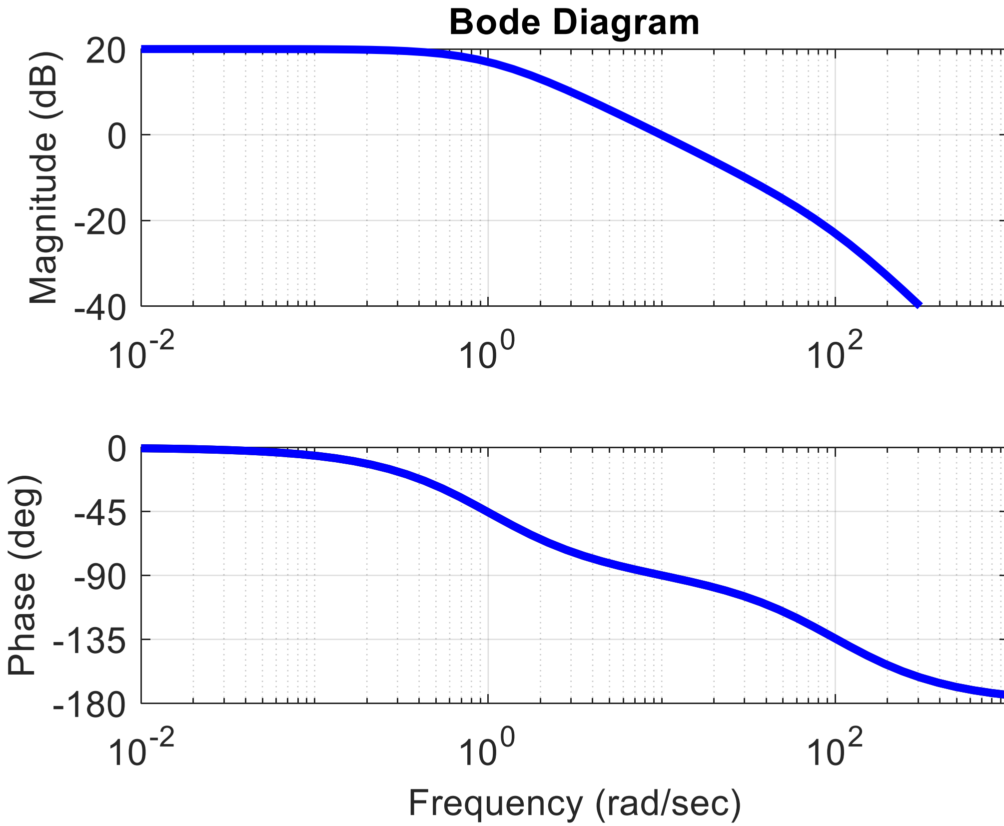


Solution 3C

C) What is the output response *y(t)* in steady-state for the input *u(t) = 2 cos(10t)*?

$$y(t) \rightarrow 2\overbrace{|G(j10)|}^{1}\cos(10t + \underbrace{\angle G(j10)}_{=-\pi/2\ \text{rads}})$$

$$2\cos(10t - \pi/2)$$



Solution 3D

D) What is the steady-state value of y(t) if the input is a unit step u(t) = 1 for all t ≥ 0?

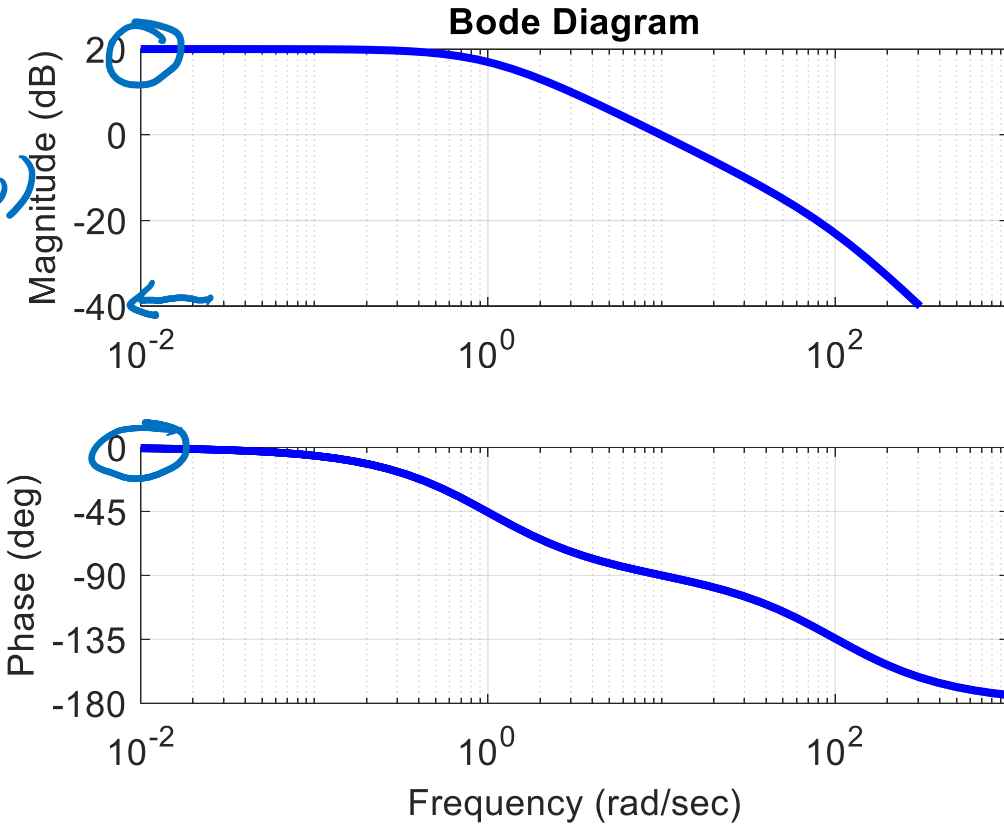
$u(t) = \cos(\omega t)$ with $\omega = 0$

$|G(j0)| = 20\,dB = 10$

$\angle G(j0) = 0\,deg = 0\,rad$

$y(t) \rightarrow \underbrace{|G(j0)|}_{10} \cos(\underset{0}{\omega} t + \underset{0}{\angle G(j0)})$

$= 10$



Solution 3-Extra Space

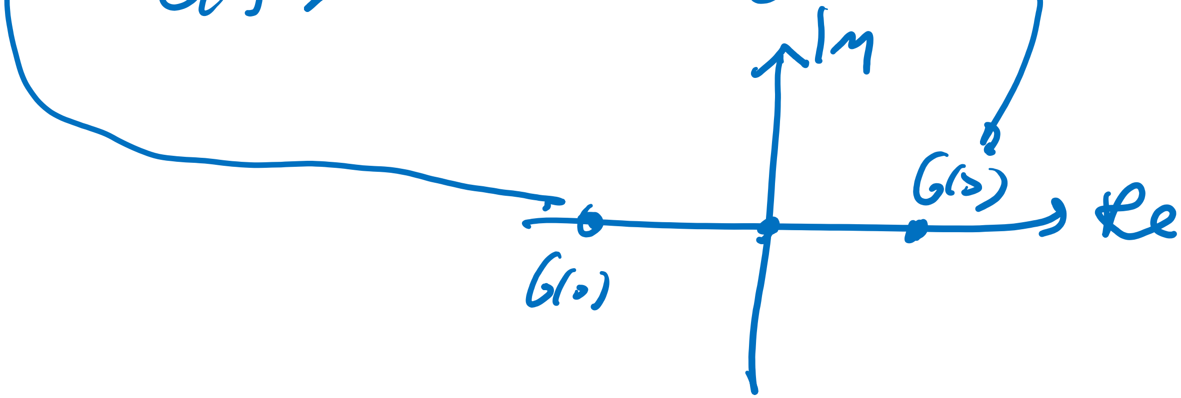
If G is stable then

$G(0)$ real

$G(0) < 0 \rightarrow \pm 180^\circ$

$G(0) = 0 \rightarrow$ undefined

$G(0) > 0 \rightarrow 0^\circ$



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Lecture 13C: Bode Plots for First-Order Systems

Key Takeaways

This lecture focuses on Bode plots for first order systems.

The Bode plot for $G(s) = \frac{b_0}{s+a_0}$ has the following key features:

- The pole defines a corner frequency ($\omega = |a_0|$) for the system.
- The magnitude is flat at low frequencies and rolls off at −20dB per decade at high frequencies.
- The phase transitions by ±90° near the corner frequency with precise details depending on the signs of (b_0, a_0).

The Bode plot for $G(s) = \frac{s+b_0}{a_0}$ has the similar features except:

- The zero defines a corner frequency ($\omega = |b_0|$) for the system.
- The magnitude rolls up at +20dB per decade at high frequencies.

Problem 4

Sketch approximate, straight-line Bode plots for the following systems:

A) $G(s) = \frac{8}{s+4}$

B) $G(s) = \frac{16}{2s-8}$

C) $G(s) = 3s + 6$

Solution 4A

A) $G(s) = \frac{8}{s+4}$

corner at $\omega = 4$ rad/sec

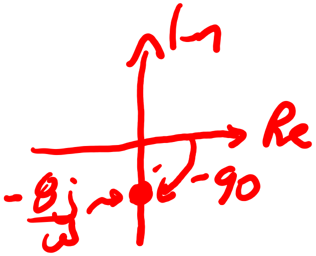
$G(0) = \frac{8}{4} = 2 \rightarrow |G(0)| = 2 = +6dB$

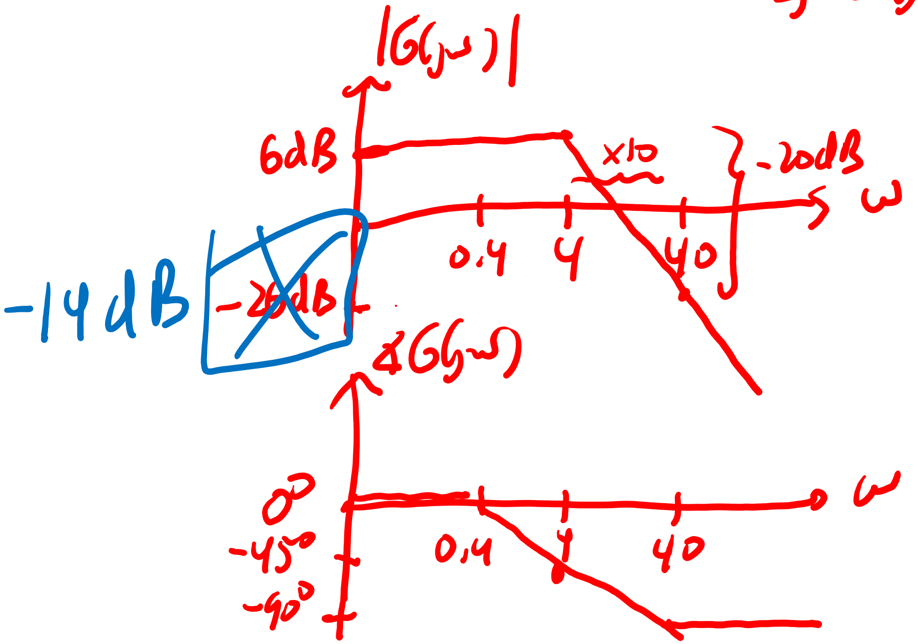
$\angle G(0) = 0°$

$\omega \rightarrow \infty \quad G(j\omega) \rightarrow \frac{8}{j\omega} \cdot \frac{j}{j} = \frac{-8j}{\omega}$

$|G(j\omega)| = |8/\omega|$

$\angle G(j\omega) = -90°$





Solution 4B

B) $G(s) = \frac{16}{2s-8}$

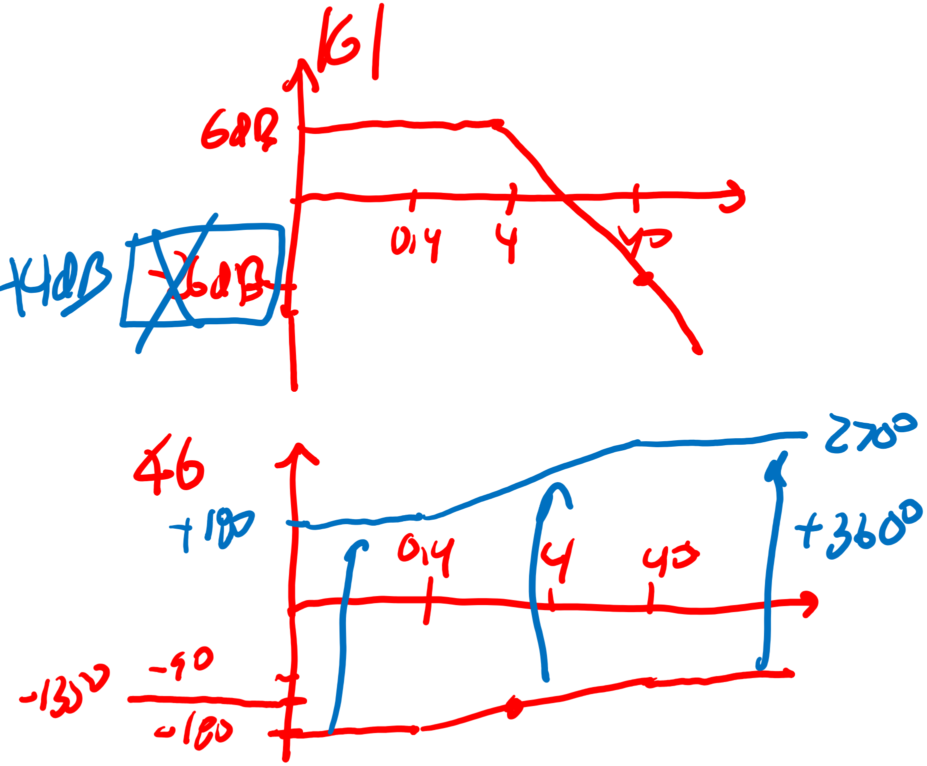
Corner $\omega = 4$ rad/sec

$G(0) = \frac{16}{-8} = -2 \rightarrow |G(j0)| = 2 = 6dB$

$\boxed{\angle G(j0) = \pm 180^\circ}$

$\omega \rightarrow \infty$, $G(j\omega) \rightarrow \frac{16}{2j\omega} \cdot \frac{j}{j} = \frac{-8j}{\omega} \rightarrow |G(j\omega)| \rightarrow 8/\omega$

$\boxed{\angle G(j\omega) \rightarrow -90^\circ \quad +270^\circ}$



Solution 4C

C) $G(s) = \underline{\underline{3s}} + 6$ $\omega = 2$ rad/sec (zero at 2 rad/sec)

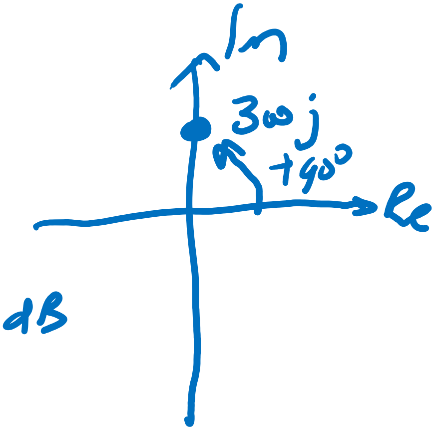
$G(0) = \underline{\underline{6}} \rightarrow |G(j0)| = 6 = 15.6$ dB

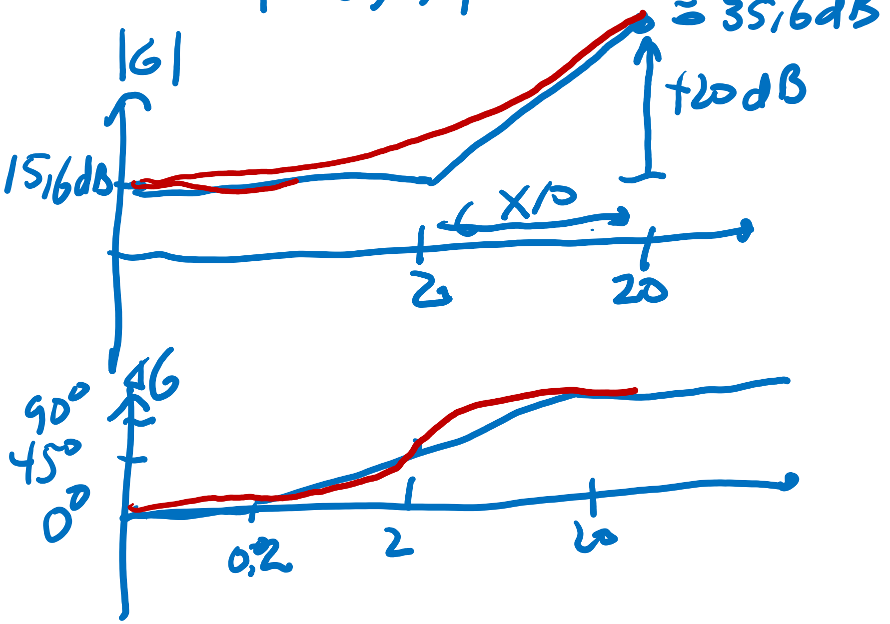
$\measuredangle G(j0) = 0°$

$\omega \rightarrow \infty$ $G(j\omega) \rightarrow 3j\omega$

$\measuredangle G(j\omega) \rightarrow +90°$

$|\cancel{\measuredangle} G(j\omega)| = 3\omega$





Solution 4-Extra Space