ECE 486: Control Systems

Lecture 13A: Steady-State Sinusoidal Response
Key Takeaways

The transfer function $G(s)$ is used to express the solution of a stable linear system forced by a sinusoidal input.

If the input is $u(t) = \sin(\omega t)$ then the response satisfies:

$$y(t) \to |G(j\omega)|\sin(\omega t + \angle G(j\omega)) \text{ as } t \to \infty$$

The output converges to a sinusoid at the same frequency as the input but with amplitude scaled by $|G(j\omega)|$ and phase is shifted by $\angle G(j\omega)$. 
The transfer function $G(s)$ was introduced as notation for an ODE.

Now we’ll think of it as a function that takes a complex number $s$ as input and returns a complex number $G(s)$.

The response of the ODE to a sinusoidal input depends on the transfer function evaluated at a purely imaginary number $s = j\omega$ where $\omega > 0$ is the frequency in rad/sec.

The result $G(j\omega)$ is a complex number that can be expressed in
- Cartesian form by its real and imaginary parts, or
- Polar form by its magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$. 
Example

- Stable, first-order system:
  \[ \dot{y}(t) + 4y(t) = 2u(t) \]
  \[ G(s) = \frac{2}{s+4} \]

- Evaluate at \( \omega = 3 \text{ rad/sec} \)
  \[ G(3j) = \frac{2}{3j+4} \cdot \frac{4-3j}{4-3j} = \frac{8-6j}{25} = 0.32 - 0.24j \]

- Cartesian Form:

- Polar Form:
  \[ |G(3j)| = \sqrt{0.32^2 + 0.24^2} = 0.4 \]
  \[ \angle G(3j) = \tan^{-1}\left(-\frac{0.24}{0.32}\right) = -0.64 \text{ rad} \]
  \[ G(j\omega) = 0.4e^{-0.64j} \]
Sinusoidal Response: First-Order Systems

Consider the stable, first-order system:
\[ y(t) + a_0 y(t) = b_0 u(t) \text{ with } y(0) = y_0 \]
\[ G(s) = \frac{b_0}{s+a_0} \]

First consider complex exponential inputs: \[ u(t) = e^{j\omega t} \]

The characteristic equation has one root: \[ s_1 = -a_0 < 0 \]

The general form of the forced-response solution is:
\[ y(t) = y_P(t) + c_1 e^{-a_0 t} \]

“Guess” the particular solution: \[ y_P(t) = c_P e^{j\omega t} \]

Sub into the ODE:
\[ j\omega c_P e^{j\omega t} + a_0 c_P e^{j\omega t} = b_0 e^{j\omega t} \quad \Rightarrow \quad c_P = \frac{b_0}{j\omega + a_0} = G(j\omega) \]

General solution:
\[ y(t) = G(j\omega)e^{j\omega t} + c_1 e^{-a_0 t} \quad \Rightarrow \quad y(t) \to G(j\omega)e^{j\omega t} \text{ as } t \to \infty \]

(Convergence depends on \( \tau_1 = \frac{1}{a_0} \))
Sinusoidal Response: First-Order Systems

Consider the stable, first-order system:

\[ \dot{y}(t) + a_0 y(t) = b_0 u(t) \quad \text{with} \quad y(0) = y_0 \]

Transfer function in polar form:

\[ G(j\omega) = |G(j\omega)|e^{j\angle G(j\omega)} \]

Recall Euler’s formula:

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \]

Take imaginary part of complex solution

\[ u(t) = e^{j\omega t} \]

\[ y(t) \rightarrow G(j\omega)e^{j\omega t} = |G(j\omega)|e^{j(\omega t + \angle G(j\omega))} \]

\[ u(t) = \sin(\omega t) \]

\[ y(t) \rightarrow |G(j\omega)|\sin(\omega t + \angle G(j\omega)) \]
Steady-State Sinusoidal Response

Consider a stable, \( n^{th} \)-order system with transfer function:

\[
G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}
\]

\[u(t) = \sin(\omega t) \quad \rightarrow \quad y(t) \rightarrow |G(j\omega)| \sin(\omega t + \angle G(j\omega))\]

\[u(t) = A \sin(\omega t + \theta) \quad \rightarrow \quad y(t) \rightarrow A|G(j\omega)| \sin(\omega t + \theta + \angle G(j\omega))\]
Example

- Consider the following stable, first-order system:
  \[ \dot{y}(t) + 4y(t) = 2u(t) \text{ with IC: } y(0) = 1.5 \quad G(s) = \frac{2}{s+4} \]
- Find response due to \( u(t) = \sin(2t) \)
- Evaluate transfer function:
  \[ \omega = \frac{2\text{ rad}}{\text{sec}} \implies |G(2j)| = 0.447 \text{ and } \angle G(2j) = -0.464 \text{ rads} \]
- Sinusoidal response:
  \[ y(t) \rightarrow 0.447 \sin(2t - 0.464) \]

Time constant is \( \tau = \frac{1}{4} \text{ sec} \)
Transient decays after
\[ 3\tau = 0.75 \text{ sec} \]
Leading vs. Lagging Response

- Steady-state sinusoidal response:
  \[ y(t) \rightarrow |G(j\omega)| \sin(\omega t + \angle G(j\omega)) \]

- Re-write as:
  \[ y(t) \rightarrow |G(j\omega)| \sin(\omega(t - t_{shift})) \text{ where } t_{shift} := -\frac{\angle G(j\omega)}{\omega} \]

- Terminology:
  - Lagging: \( \angle G(j\omega) < 0 \Rightarrow t_{shift} > 0 \)
  - Leading: \( \angle G(j\omega) > 0 \Rightarrow t_{shift} < 0 \)

  \[ y(t) \rightarrow 0.447 \sin(2t - 0.464) \]

  \[ y(t) \rightarrow 0.447 \sin(2(t - 0.232)) \]

  \[ u(2\pi) = 0 \text{ and } y(t) = 0 \text{ at } t = 2\pi + 0.232 \approx 6.515 \text{ sec.} \]