ECE486: Control Systems

▶ Lecture 11A: Introduction to Root Locus design method

Goal: introduce the Root Locus method as a way of visualizing the locations of closed-loop poles of a given system as some parameter is varied.

Reading: FPE, Chapter 5
The Root Locus Design Method
(invented by Walter R. Evans in 1948)

Consider this unity feedback configuration:

\[ R \rightarrow + \rightarrow K \rightarrow L(s) \rightarrow Y \]

where

- \( K \) is a constant gain
- \( L(s) = \frac{b(s)}{a(s)} \), where \( a(s) \) and \( b(s) \) are some polynomials

**Problem:** How to choose \( K \) to stabilize the closed-loop system?
The Root Locus Design Method

Closed-loop transfer function: \( \frac{Y}{R} = \frac{KL(s)}{1 + KL(s)} \), \( L(s) = \frac{b(s)}{a(s)} \)

Closed loop poles are solutions of:

\[
1 + KL(s) = 0 \quad \Leftrightarrow \quad L(s) = -\frac{1}{K}
\]

\[
1 + \frac{Kb(s)}{a(s)} = 0
\]

\[
\underbrace{a(s) + Kb(s)} = 0 \quad \text{characteristic polynomial}
\]
Note the change of notation:

\[
G(s) = \frac{q(s)}{p(s)} \quad \text{to} \quad L(s) = \frac{b(s)}{a(s)}
\]

— the RL method is quite general, so \(L(s)\) is not necessarily the plant transfer function, and \(K\) is not necessary feedback gain (could be any parameter).

E.g., \(L(s)\) and \(K\) may be related to plant transfer function and feedback gain through some transformation.

As long as we can represent the poles of the closed-loop transfer function as roots of the equation \(1 + KL(s) = 0\) for some choice of \(K\) and \(L(s)\), we can apply the RL method.
Towards Quantitative Characterization of Stability

Qualitative description of stability: Routh test gives us a range of $K$ to guarantee stability.

For what values of $K$ do we best satisfy given design specs?
Root Locus and Quantitative Stability

Closed-loop transfer function: \[ \frac{Y}{R} = \frac{KL(s)}{1 + KL(s)} \], \[ L(s) = \frac{b(s)}{a(s)} \]

For what values of \( K \) do we best satisfy given design specs?

Specs are encoded in pole locations, so:

The \textit{root locus} for \( 1 + KL(s) \) is the set of all closed-loop poles, i.e., the roots of

\[ 1 + KL(s) = 0, \]

as \( K \) varies from 0 to \( \infty \).
A Simple Example

\[ L(s) = \frac{1}{s^2 + s} \]
\[ b(s) = 1, \ a(s) = s^2 + s \]

Characteristic equation:
\[ a(s) + Kb(s) = 0 \]
\[ s^2 + s + K = 0 \]

Here, we can just use the quadratic formula:
\[ s = -\frac{1 \pm \sqrt{1 - 4K}}{2} = -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2} \]

Root locus = \( \left\{ -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2} \ : 0 \leq K < \infty \right\} \subset \mathbb{C} \)
Example, continued

$$\text{Root locus} = \left\{ -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2} : 0 \leq K < \infty \right\} \subset \mathbb{C}$$

Let’s plot it in the $s$-plane:

- start at $K = 0$ the roots are $-\frac{1}{2} \pm \frac{1}{2} \equiv -1, 0$

Note: these are poles of $L$ (open-loop poles)
Example, continued

Root locus: \[ \left\{ -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2} : 0 \leq K < \infty \right\} \subset \mathbb{C} \]

- as \( K \) increases from 0, the poles start to move

\[ 1 - 4K > 0 \implies 2 \text{ real roots} \]

\[ K = 1/4 \implies 1 \text{ real root } s = -1/2 \]
Example, continued

Root locus: \[ \left\{ -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2} : 0 \leq K < \infty \right\} \subset \mathbb{C} \]

- as \( K \) increases from 0, the poles start to move

\[ K > \frac{1}{4} \quad \implies \quad 2 \text{ complex roots with } \text{Re}(s) = -\frac{1}{2} \]

\( s = -1/2 \) is the point of breakaway from the real axis.
Example, continued

Compare this to admissible regions for given specs:

\[ t_s \approx \frac{3}{\sigma} \quad \text{want } \sigma \text{ large, can only have } \sigma = \frac{1}{2} \quad (t_s = 6) \]

\[ t_r \approx \frac{1.8}{\omega_n} \quad \text{want } \omega_n \text{ large } \implies \text{ want } K \text{ large} \]

\[ M_p \quad \text{want to be inside the shaded region } \implies \text{ want } K \text{ small} \]
Thus, the root locus helps us visualize the trade-off between all the specs in terms of $K$.

However, for order $> 2$, there will generally be no direct formula for the closed-loop poles as a function of $K$.

Our goal: develop simple rules for (approximately) sketching the root locus in the general case.