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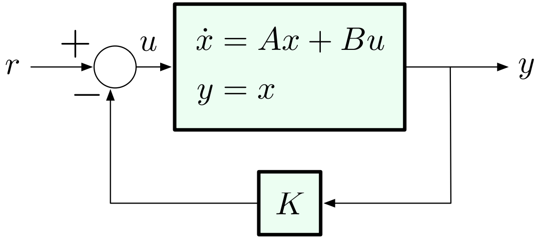
Reading: FPE, Chapter 7

Review: Pole Placement via State Feedback

Assume that the plant is controllable:



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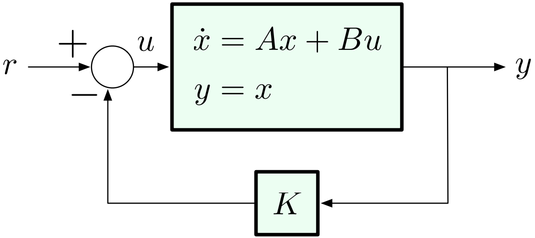

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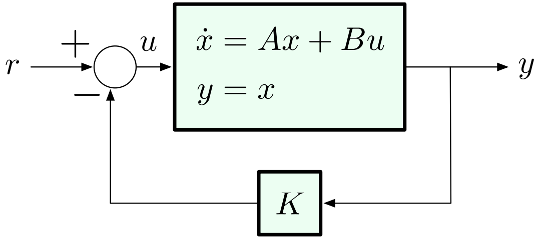

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$$Y(s) = (Is - A + BK)^{-1} BR(s)$$

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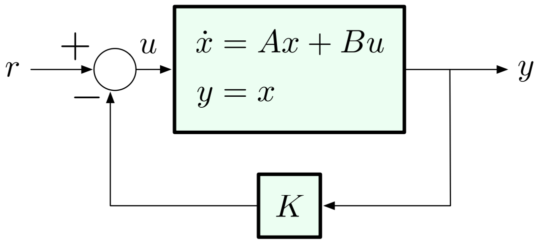

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Closed-loop poles are the eigenvalues of $A - BK$!!

Review: Pole Placement in CCF

$$\dot{x} = (A - BK)x + Br, \qquad y = Cx$$

$$A - BK = -\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ a_n + k_1 & a_{n-1} + k_2 & \dots & a_2 + k_{n-1} & a_1 + k_n \end{pmatrix}$$

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$$\det(Is - A + BK)$$
$$= s^n + (a_1 + k_n)s^{n-1} + \ldots + (a_{n-1} + k_2)s + (a_n + k_1)$$

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Key observation: When the system is in CCF, each control gain affects only *one* of the coefficients of the characteristic polynomial, and these coefficients can be assigned arbitrarily by a suitable choice of $k_1, \ldots, k_n$.

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Hence the name Controller Canonical Form — convenient for control design.

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General procedure for any *controllable* system:

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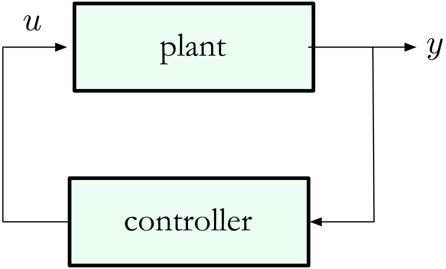
3. Convert back to original coordinates.

Is Full State Feedback Always Available?

In a typical system, measurements are provided by sensors:



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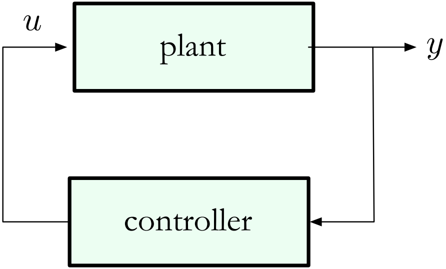

Full state feedback $u = -Kx$ is *not implementable*!!

When Full State Feedback Is Unavailable ...

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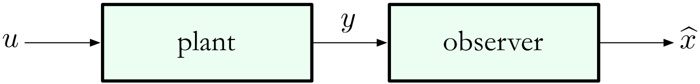
State Estimation Using an Observer

When full state feedback is unavailable, the observer is used to estimate the state x:



State Estimation Using an Observer

The idea is to design the observer in such a way that the state estimate $\widehat{x}$ is *asymptotically accurate*:

$$\|\widehat{x}(t) - x(t)\| = \sqrt{\sum_{i=1}^{n} \left(\widehat{x}_i(t) - x_i(t)\right)^2} \xrightarrow{t \to \infty} 0$$

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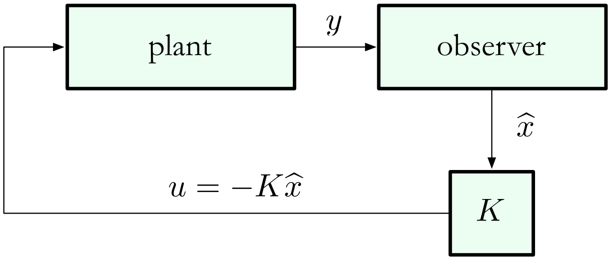
If we are successful, then we can try estimated state feedback:



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- Before, we saw that closed-loop poles can be assigned arbitrarily by full state feedback when the plant is controllable.
- Now, we will see that asymptotically accurate state estimation will be possible when the system is observable.
- Observability is a system property which is dual to controllability.

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$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

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(This definition is only true for the single-output case; the multiple-output case involves the *rank* of $\mathcal{O}(A, C)$.)

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— recall: this system is in Observer Canonical Form (OCF) ...

Observer Canonical Form

A single-output state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Observer Canonical Form (OCF) if the matrices A, C are of the form

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(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

Coordinate Transformations and Observability

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If the original system is observable, then

$$T \underbrace{\left[\mathcal{O}(A, C)\right]^{-1}}_{\text{old}} = \underbrace{\left[\mathcal{O}(\bar{A}, \bar{C})\right]^{-1}}_{\text{new}}$$

$$\Updownarrow$$

$$T = \underbrace{\left[\mathcal{O}(\bar{A}, \bar{C})\right]^{-1}}_{\text{new}} \underbrace{\left[\mathcal{O}(A, C)\right]}_{\text{old}}$$

Observability and State Estimation

As we will show next:

If the system is observable, then there exists an observer (state estimator) that provides an asymptotically convergent estimate $\widehat{x}$ of the state x based on the observed output y.

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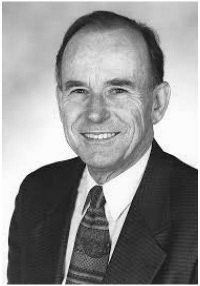

The particular type of observer we will construct is called the Luenberger observer after David G. Luenberger, who developed this idea in his 1963 Ph.D. dissertation.

David Luenberger is a Professor at Stanford University.

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At this point, we do not assume anything about observability.

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$$\begin{aligned} \text{System:} \qquad & \dot{x} = Ax \\ & y = Cx \\ \text{Observer:} \qquad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly. \end{aligned}$$

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What happens to state estimation error $e = x - \widehat{x}$ as $t \to \infty$?

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Does $e(t)$ converge to zero in some sense?

Linear ODEs and Eigenvalues: A Digression

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If all λ_i's have negative real parts, then

$$\begin{aligned} \|v(t)\|^2 &= v(t)^T v(t) = \bar{v}(t)^T \bar{v}(t) \\ &\leq Ce^{-2\sigma_{\min} t}, \qquad \text{where } \sigma_{\min} = \min_{1 \leq i \leq n} |\mathrm{Re}(\lambda_i)| \end{aligned}$$

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Recall our assumption that $A - LC$ is Hurwitz (all eigenvalues are in LHP). This implies that

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\|x(t) - \widehat{x}(t)\|^2 = \|e(t)\|^2 = \sum_{i=1}^{n} |e_i(t)|^2 \xrightarrow{t\to\infty} 0
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For fast convergence, want eigenvalues of $A - LC$ far into LHP!!

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Observer transfer function:

$$\begin{aligned} & s\widehat{X}(s) = (A - LC)\widehat{X}(s) + LY(s) \\ & (Is - A + LC)\widehat{X}(s) = LY(s) \\ & \widehat{X}(s) = (Is - A + LC)^{-1}LY(s). \end{aligned}$$

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\end{aligned}$$

The eigenvalues of $A - LC$ are the observer poles. We want these poles to be *stable* and *fast*.

Observability and Estimation Error

Fact: If the system

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is observable, then we can arbitrarily assign eigenvalues of $A - LC$ by a suitable choice of the output injection matrix L.

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This is similar to the fact that controllability implies arbitrary closed-loop pole placement by state feedback.

In fact, these two facts are closely related because CCF is dual to OCF.

Observer Pole Placement in OCF

Consider a single-output system in OCF:

$$
\begin{aligned}
\dot{x} &= Ax \\
y &= Cx, \qquad y \in \mathbb{R}
\end{aligned}
$$

where $A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -a_n \\ 1 & 0 & \dots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -a_2 \\ 0 & 0 & \dots & 0 & 1 & -a_1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}$

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Note that A^T has the form of a CCF system matrix, thus:

$$\begin{aligned} \det(Is - A) &= \det((Is - A)^T) = \det(Is - A^T) \\ &= s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \end{aligned}$$

Now Let's Add an Observer

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_n \\ 0 & 1 & \dots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & \dots & 1 & -a_1 \end{pmatrix}$$

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— still in OCF!!

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Eigenvalues of $A - LC$ are the roots of the characteristic polynomial

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Key observation: In OCF, each observer gain affects only *one* of the coefficients of the characteristic polynomial, which can be assigned arbitrarily by a suitable choice of $\ell_1, \ldots, \ell_n$.

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Hence the name Observer Canonical Form — convenient for observer design.

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In fact, this procedure is not necessary because of duality between controllability and observability!!

Controllability–Observability Duality

Claim: The system

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is observable if and only if the system

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Thus, $\mathcal{O}(A, C)$ is nonsingular if and only if $\mathcal{C}(A^T, C^T)$ is.

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Final answer: use the observer

$$\begin{aligned}\dot{\widehat{x}} &= (A - LC)\widehat{x} + Ly \\ &= (A - K^T C)\widehat{x} + K^T y.\end{aligned}$$

Combining Full-State Feedback with an Observer

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- So far, we have focused on autonomous systems ($u = 0$).
- What about nonzero inputs?

$$\dot{x} = Ax + Bu$$
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