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- Review: stability from frequency response
- Today's topic: control design using frequency response

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Reading: FPE, Chapter 6

Review: Phase Margin for 2nd-Order System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}, \qquad \text{closed-loop t.f.} \ = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\text{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{4\zeta^4 + 1} - 2\zeta^2}\right) \approx 100 \cdot \zeta$$

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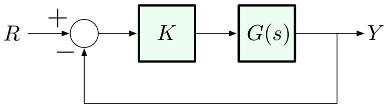
Thus, the overshoot $M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ and resonant peak $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$ are both related to PM through ζ!!

Bode's Gain-Phase Relationship



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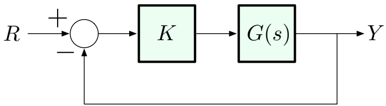

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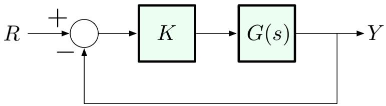

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	low freq.	real zero/pole	complex zero/pole
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phase	$n \times 90°$	up/down by 90°	up/down by 180°

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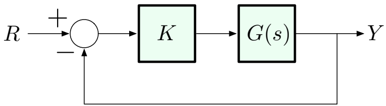

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We can state this succinctly as follows:

Gain-Phase Relationship. Far enough from break-points,

$$\text{Phase} \approx \text{Magnitude Slope} \times 90°$$

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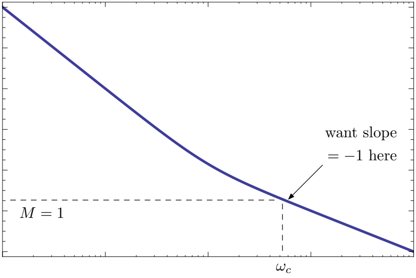

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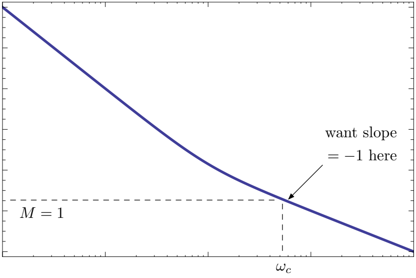

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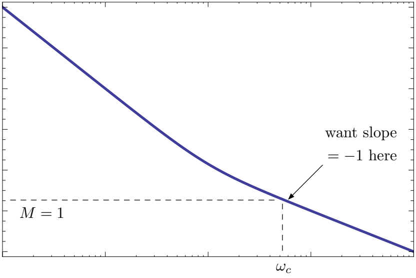

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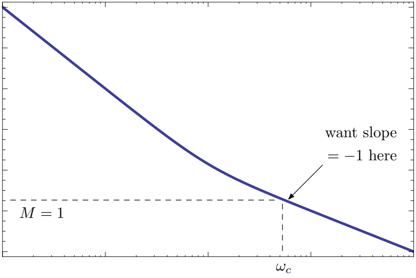

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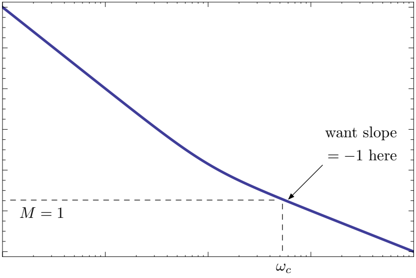

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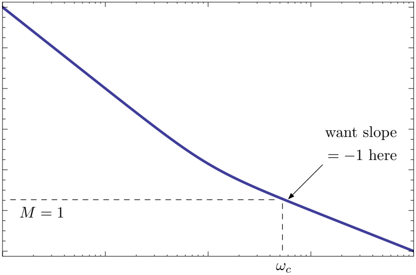

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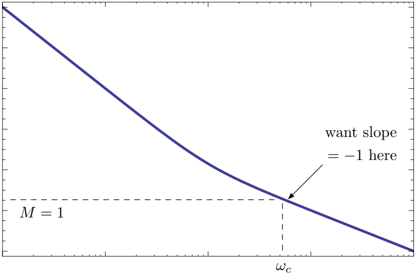

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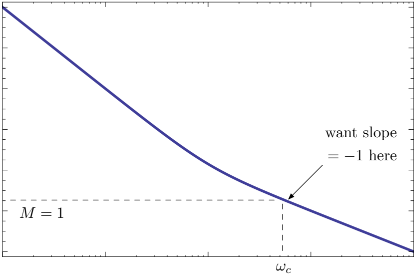

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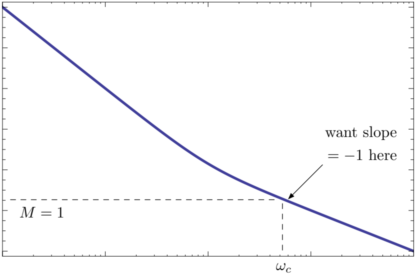

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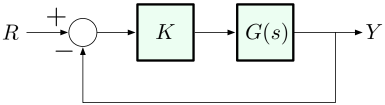
(Similar considerations apply when M-plot has positive slope – depends on the t.f.)

Gain-Phase Relationship & Bandwidth



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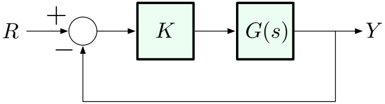
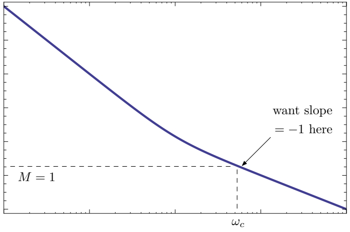


M-plot for *open-loop* t.f. KG:



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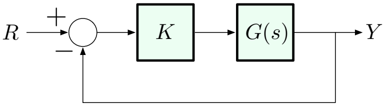

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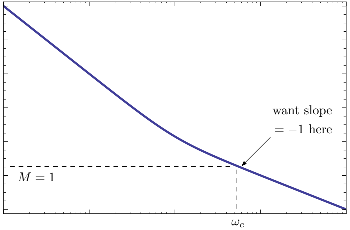

Note: $|KG(j\omega)| \to \infty$ as $\omega \to 0$

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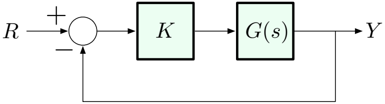

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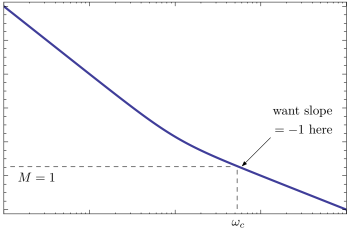

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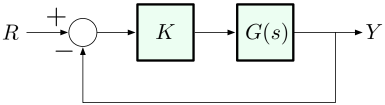

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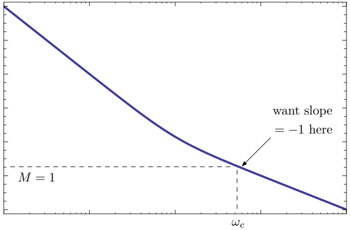

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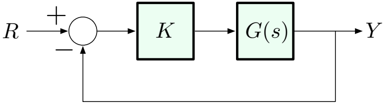

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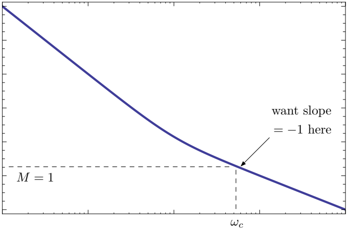

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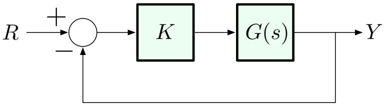

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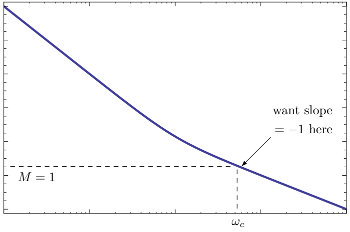

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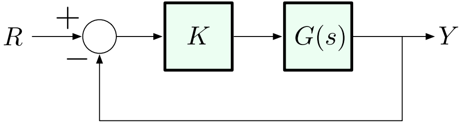
- If PM $= 90^\circ$, then $\omega_c = \omega_{\text{BW}}$
- If $PM < 90^\circ$, then $\omega_c \leq \omega_{\text{BW}} \leq 2\omega_c$ (see FPE)

Control Design Using Frequency Response



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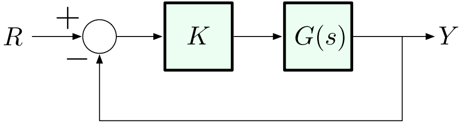

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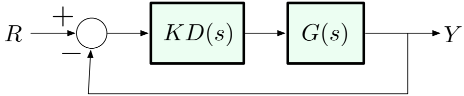
In particular, from the quantitative Gain-Phase Relationship,

$$\text{Magnitude slope}(\omega_c) = -1 \qquad \Longrightarrow \qquad \text{Phase}(\omega_c) \approx -90^\circ$$

— which gives us PM of 90° and consequently good damping.

Example



Let $G(s) = \dfrac{1}{s^2}$ (double integrator)

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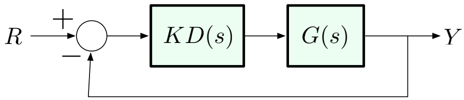

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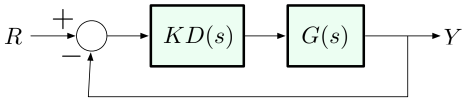

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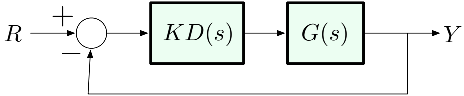

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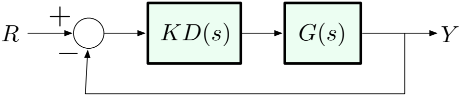

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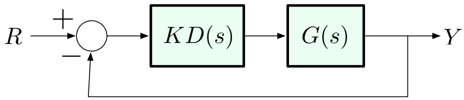

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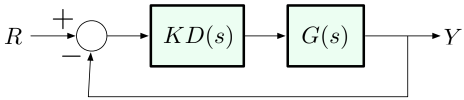

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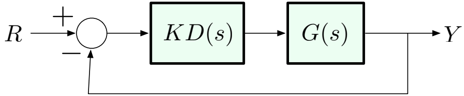

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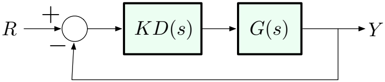
- if $\mathrm{PM} = 90^\circ$, then $\omega_c = \omega_{\mathrm{BW}} \Longrightarrow$ want $\omega_c \approx 0.5$

Design, First Attempt



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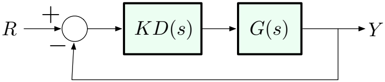

$$G(s) = \frac{1}{s^2}$$

Let's try proportional feedback:

$$D(s) = 1 \implies KD(s)G(s) = KG(s) = \frac{K}{s^2}$$

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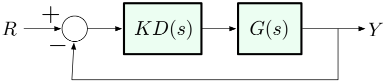

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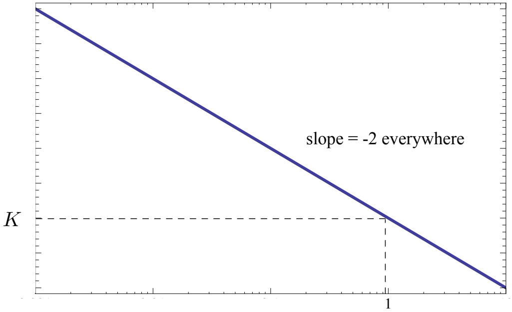

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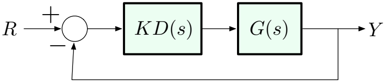

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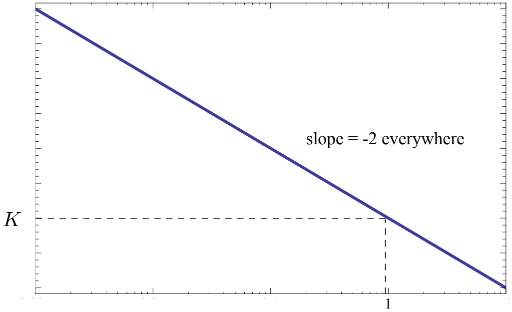

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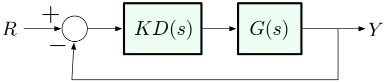

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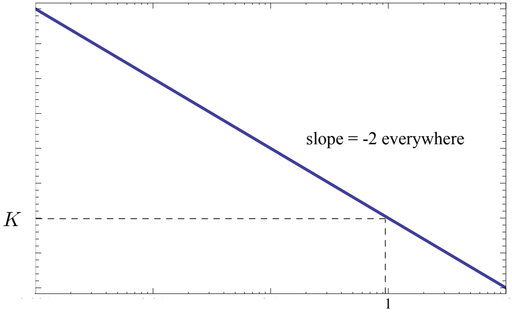

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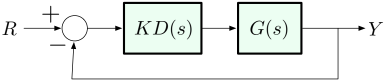
We already know that P-gain alone won't do the job:

$$K + s^2 = 0 \text{ (imag. poles)}$$

Design, Second Attempt



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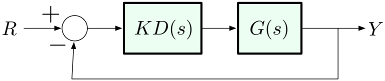

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$$KD(s) = K(\tau s + 1), \qquad \text{where } K = K_{\mathrm{P}},\ K\tau = K_{\mathrm{D}}$$

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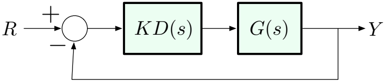

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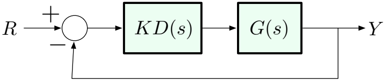

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Bode plot interpretation: PD controller introduces a Type 2 term in the numerator, which pushes the slope up by 1

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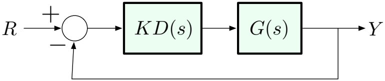

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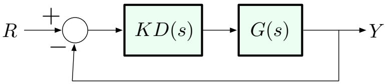
— this has the effect of pushing the M-slope of $KD(s)G(s)$ from -2 to -1 past the break-point ($\omega = 1/\tau$).

Design, Second Attempt (PD-Control)



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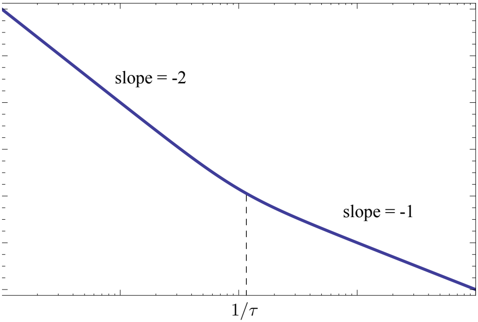

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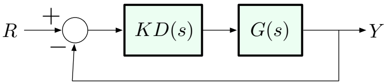

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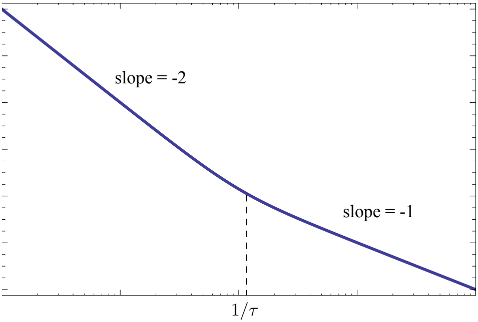

For the G-P relationship to be valid, choose the break-point several times smaller than desired ω_c:

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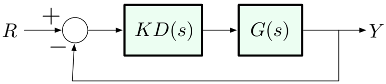

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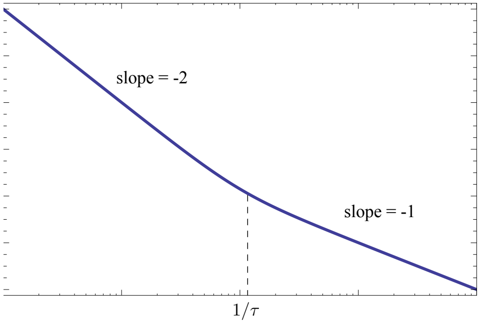

For the G-P relationship to be valid, choose the break-point several times smaller than desired ω_c:
$\Longrightarrow$ let's take $\tau = 10$

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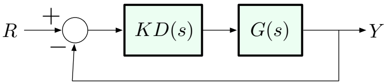

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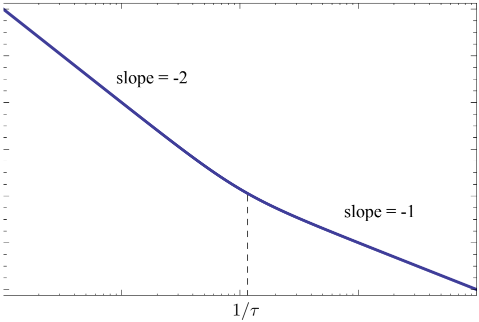

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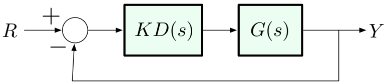

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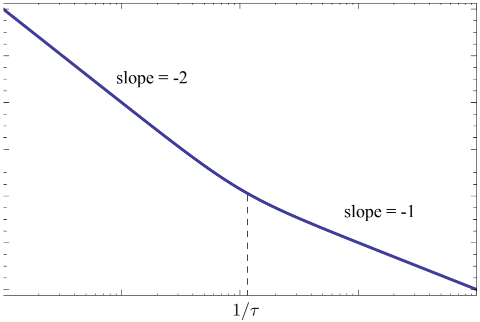

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Open-loop t.f.:

$$KD(s)G(s) = \frac{K(10s + 1)}{s^2}$$

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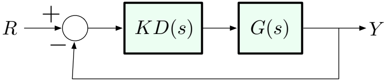

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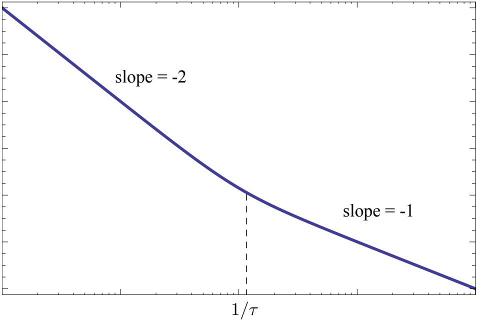

- Want $\omega_c \approx 0.5$

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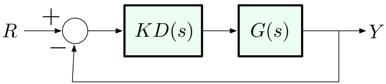

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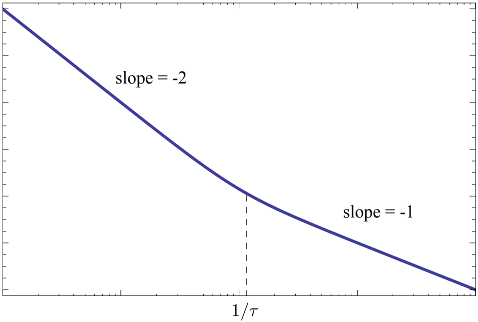
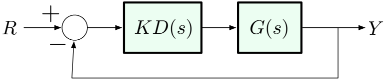


- Want $\omega_c \approx 0.5$
- This means that

$$\begin{aligned} M(j0.5) &= 1 \\ |KD(j0.5)G(j.05)| & \\ &= \frac{K|5j+1|}{0.5^2} \\ &= 4K\sqrt{26} \approx 20K \end{aligned}$$

$$\Longrightarrow K = \frac{1}{20}$$

PD Control Design: Evaluation



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Initial design: $KD(s) = \dfrac{10s + 1}{20}$

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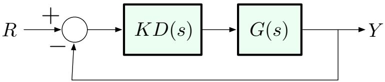

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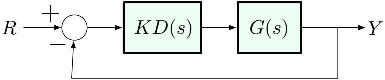

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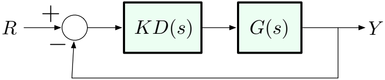

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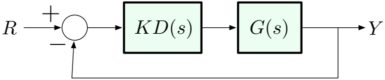

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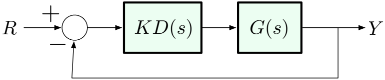

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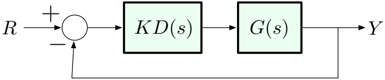

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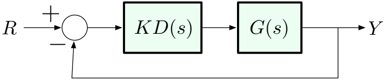

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- PD control increases slope $\longrightarrow$ increases $\omega_c \longrightarrow$ increases $\omega_{\text{BW}} \longrightarrow$ faster response
- usual complaint: D-gain is not physically realizable, so let's try lead compensation

Lead Compensation: Bode Plot

$$KD(s) = K\frac{s+z}{s+p}, \qquad p \gg z$$

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Break-points:

- Type 1 zero with break-point at $\omega = z$ (comes first, $z \ll p$)

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Break-points:

- Type 1 zero with break-point at $\omega = z$ (comes first, $z \ll p$)
- Type 1 pole with break-point at $\omega = p$

Lead Compensation: Bode Plot

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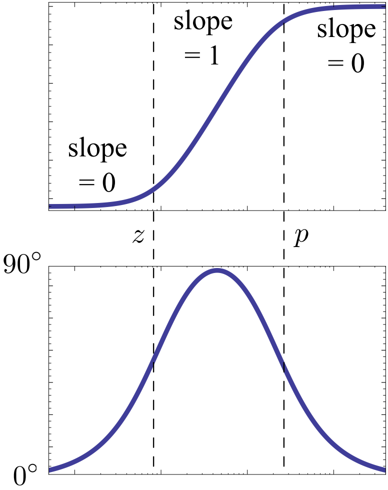

- magnitude levels off at high frequencies $\Longrightarrow$ better noise suppression

- adds phase, hence the term "phase lead"

Lead Compensation and Phase Margin

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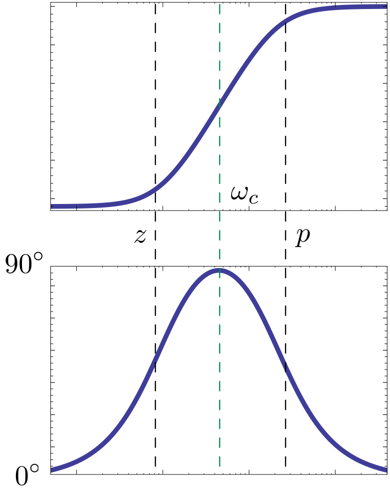

For best effect on PM, ω_c should be halfway between z and p (on log scale):

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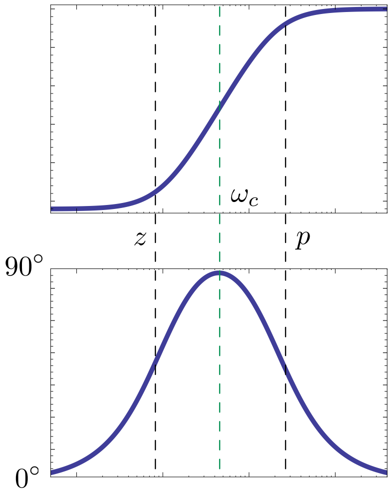

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$$\text{or } \omega_c = \sqrt{z \cdot p}$$

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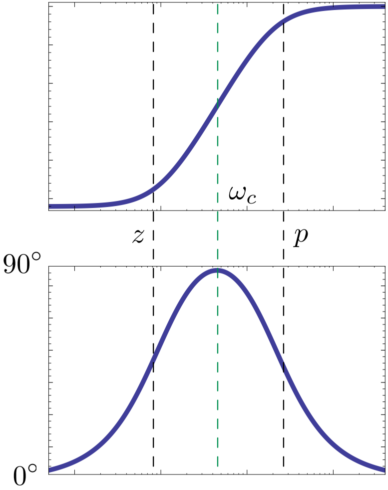

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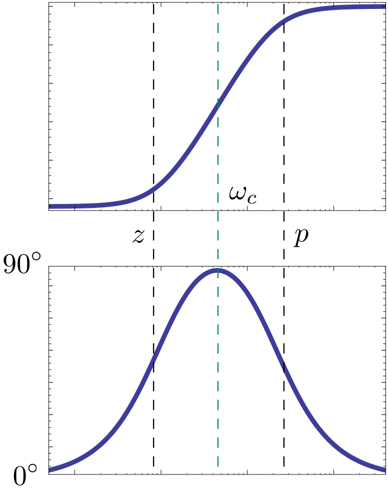

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Trade-offs: large $p - z$ means

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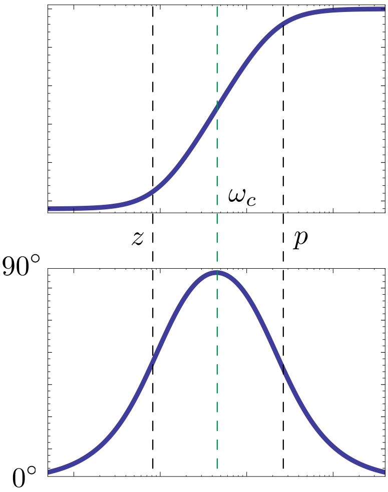

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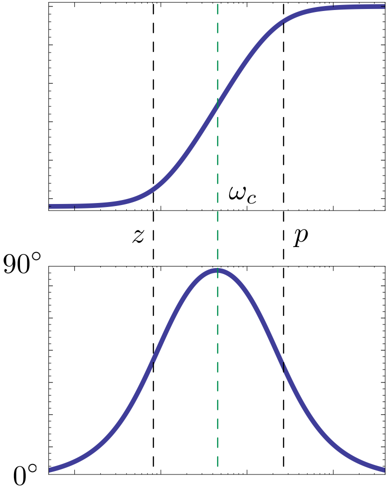

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— geometric mean of z and p

Trade-offs: large $p - z$ means

- large PM (closer to 90°)
- but also bigger M at higher frequencies (worse noise suppression)

Back to Our Example: $G(s) = \dfrac{1}{s^2}$

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Objectives (same as before):

- stability
- good damping
- ω_{BW} close to 0.5

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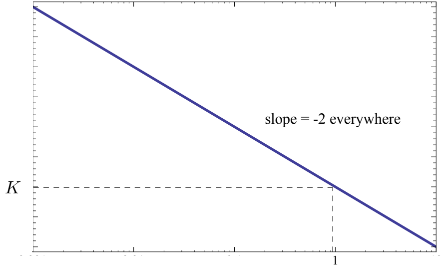
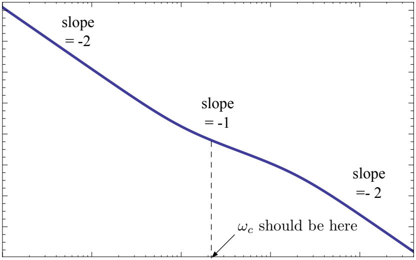
$KG(s) = \frac{K}{s^2}$ (w/o lead):



$$\frac{K}{(0.5)^2} = 1 \implies K = \frac{1}{4}$$

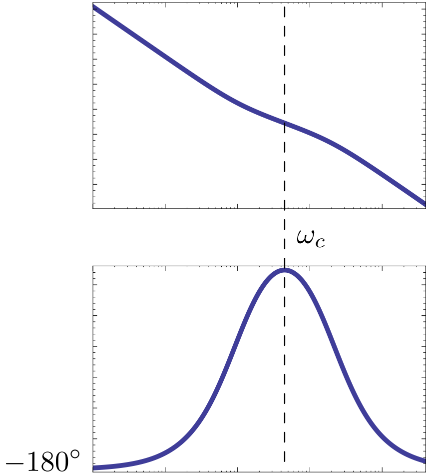
after adding lead:



— adding lead will increase ω_c!!

Back to Our Example: $G(s) = \frac{1}{s^2}$

After adding lead with $K = 1/4$, what do we see?



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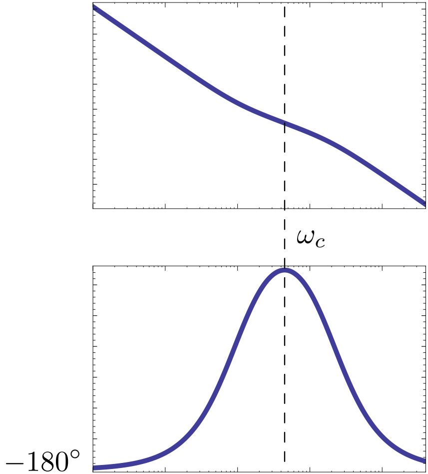

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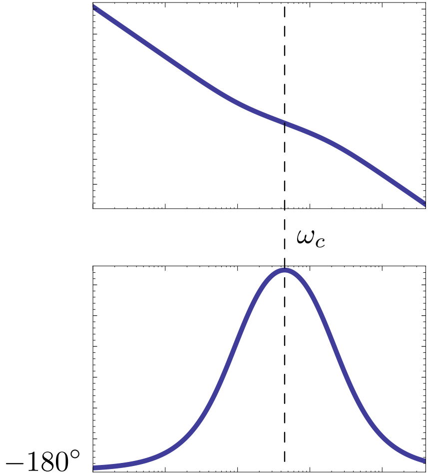

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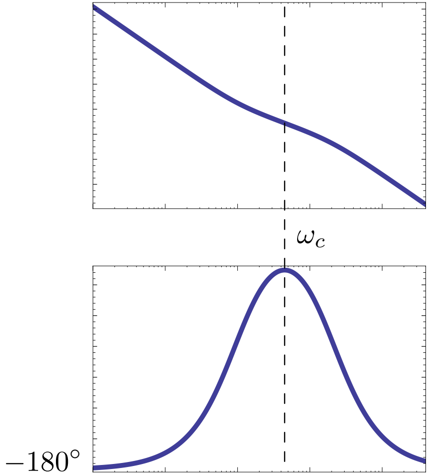

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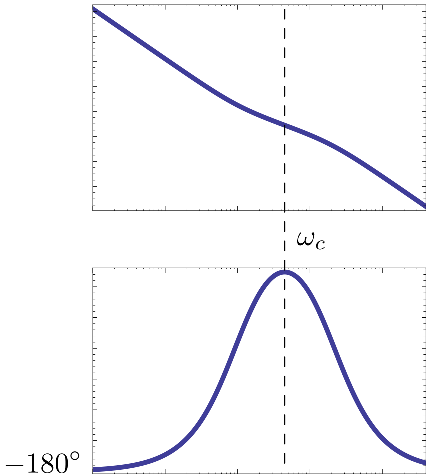

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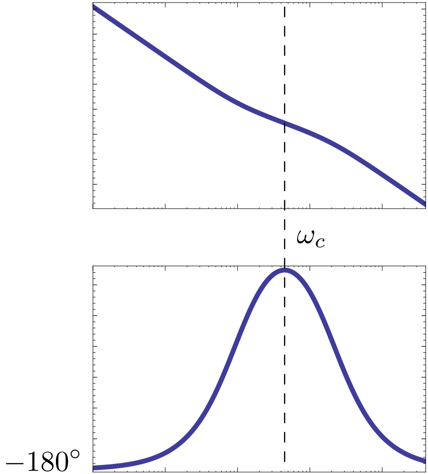

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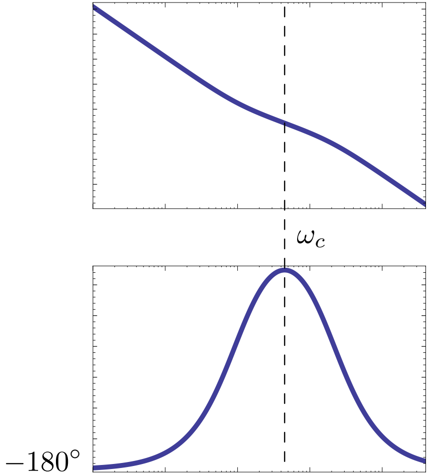

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Thus, we want

$$\omega_c = 0.25 \implies K = \frac{1}{16}$$

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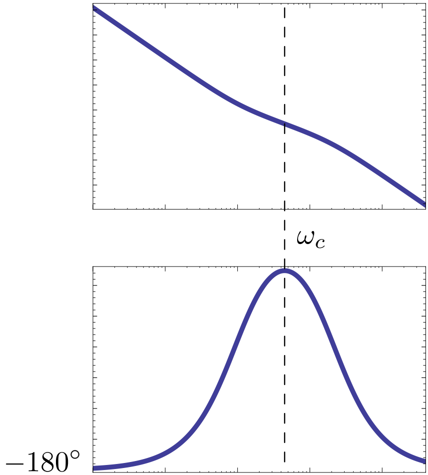

Next, we pick z and p so that ω_c is approximately their geometric mean:

e.g., $z = 0.1$, $p = 2$

$$\sqrt{z \cdot p} = \sqrt{0.2} \approx 0.447$$

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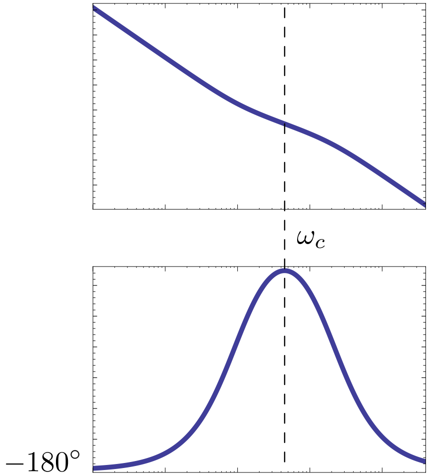

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Resulting lead controller:

$$KD(s) = \frac{1}{16} \frac{\frac{s}{0.1} + 1}{\frac{s}{2} + 1}$$

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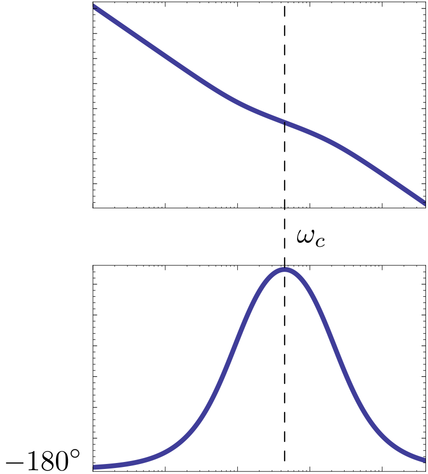

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(may still need to be refined using Matlab)

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This is an intuitive procedure, but it's not very precise, requires trial & error.