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- Review: dynamic response; transfer functions; transient and steady-state response
- Today's topic: dynamic response (transient and steady-state) with arbitrary I.C.'s

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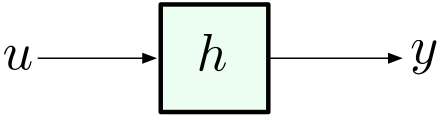
Reading: FPE, Section 3.1, Appendix A

Dynamic Response



Problem: compute the response y to a given input u under a given set of initial conditions.

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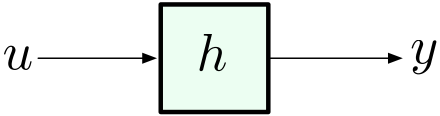

Problem: compute the response y to a given input u under a given set of initial conditions.

In particular, we wish to know both the transient response (due to I.C.'s) and the steady-state response (once the effect of the I.C.'s "washes away").

Laplace Transforms Revisited

(see FPE, Appendix A)

One-sided (or unilateral) Laplace transform:

$$\mathscr{L}\{f(t)\} \equiv F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t \qquad \text{(really, from } 0^-\text{)}$$

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$$f(t) = 1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

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— this is valid provided $\mathrm{Re}(s) > 0$, so that $e^{-st} \xrightarrow{t\to+\infty} 0$.

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— in both cases, require $\mathrm{Re}(s) > 0$, i.e., s must lie in the right half-plane (RHP)

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for $\text{Re}(s) > 0$

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— can't find $Y(s)$ in the tables. So how do we compute y?

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\left\{\dfrac{s}{(s+1)(s^2+1)}\right\}$

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$$\frac{s}{s^2+1} \qquad \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t \qquad (\#18)$$

— so we see some things that are similar to $Y(s)$, but not quite.

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This brings us to the method of partial fractions:

- boring (i.e., character-building), but *very useful*
- allows us to break up complicated fractions into sums of simpler ones, for which we know $\mathscr{L}^{-1}$ from tables

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We seek a, b, c, such that

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$$(s+1)Y(s) = \frac{s}{s^2+1} = a + \frac{(s+1)(as+b)}{(s^2+1)}$$

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$$(s+1)Y(s) = \frac{s}{s^2+1} = a + \frac{(s+1)(as+b)}{(s^2+1)}$$

— now let $s = -1$ to “kill” the second term on the RHS:

$$a = (s+1)Y(s)\Big|_{s=-1} = -\frac{1}{2}$$

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Match $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ parts:

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&= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \quad \text{(from tables)} \\
&= -\frac{1}{2}e^{-t} + \frac{1}{\sqrt{2}}\cos(t - \pi/4) \quad (\cos(a-b) = \cos a \cos b + \sin a \sin b)
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— the freq. response formula gives only the steady-state part!!

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Therefore, the first formula is correct.

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Main message: the frequency response formula only gives the steady-state part of the response, but the inverse Laplace transform gives the whole response (including the transient part).

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— we will now see how to deal with nonzero I.C.'s ...

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Similarly:

$$\begin{aligned}\mathscr{L}\{f''(t)\} = \mathscr{L}\{(f'(t))'\} &= s\mathscr{L}\{f'(t)\} - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

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$$s^2 Y(s) - s\alpha - \beta + 3sY(s) - 3\alpha + 2Y(s) = U(s)$$

$U(s) = \mathscr{L}\{1(t)\} = 1/s$, which gives

$$s^2 Y(s) - s\alpha - \beta + 3sY(s) - 3\alpha + 2Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{\alpha s + (3\alpha + \beta) + \frac{1}{s}}{s^2 + 3s + 2} = \frac{\alpha s^2 + (3\alpha + \beta)s + 1}{s(s+1)(s+2)}$$

Note: if $\alpha = \beta = 0$, then $Y(s) = \dfrac{1}{s(s+1)(s+2)} = H(s)U(s)$

Example (continued)

Compute the step response of

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \alpha,\ \dot{y}(0) = \beta$$

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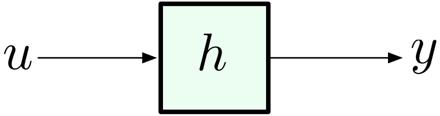
DC Gain



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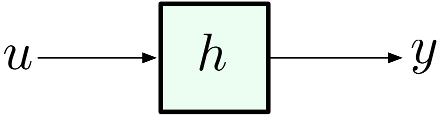

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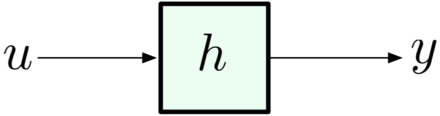

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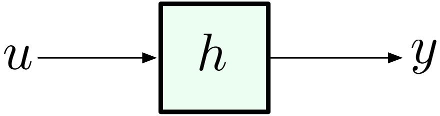

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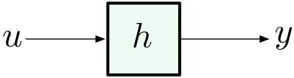
therefore, DC gain $= y(\infty) = 1/2$

Steady-State Value



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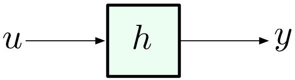

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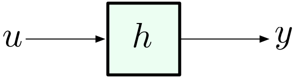

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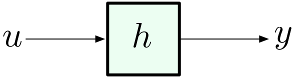

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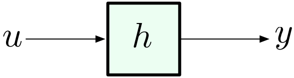

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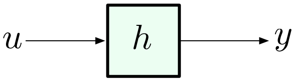

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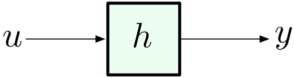

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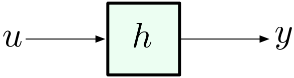

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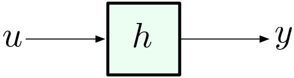

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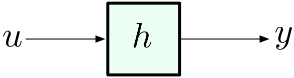

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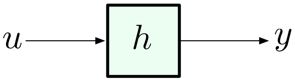

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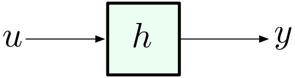

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If all poles of $sY(s)$ are *strictly stable* or lie in the *open left half-plane* (OLHP), i.e., have $\mathrm{Re}(s) < 0$, then

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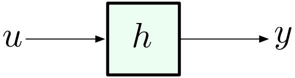
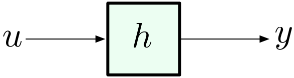
- $Y(s) = \dfrac{1}{s^2+\omega^2}$ $\qquad sY(s) = \dfrac{s}{s^2+\omega^2}$
  poles are purely imaginary (not in OLHP), FVT does not give correct answer
- $Y(s) = \dfrac{c}{s}$ $\qquad sY(s) = c$
  poles at infinity, so $y(\infty) = c$ – FVT gives correct answer

Back to DC Gain

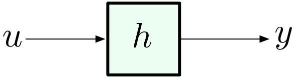


Back to DC Gain



Step response: $$Y(s) = \frac{H(s)}{s}$$

Back to DC Gain



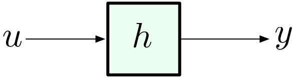
Step response: $$Y(s) = \frac{H(s)}{s}$$

— if all poles of $sY(s) = H(s)$ are strictly stable, then

$$y(\infty) = \lim_{s \to 0} H(s)$$

by the FVT.

Back to DC Gain



$$\text{Step response:} \qquad Y(s) = \frac{H(s)}{s}$$

— if all poles of $sY(s) = H(s)$ are strictly stable, then

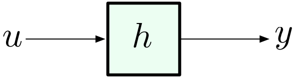
$$y(\infty) = \lim_{s \to 0} H(s)$$

by the FVT.

Example: compute DC gain of the system with transfer function

$$H(s) = \frac{s^2 + 5s + 3}{s^3 + 4s + 2s + 5}$$

Back to DC Gain



$$\text{Step response:} \qquad Y(s) = \frac{H(s)}{s}$$

— if all poles of $sY(s) = H(s)$ are strictly stable, then

$$y(\infty) = \lim_{s \to 0} H(s)$$

by the FVT.

Example: compute DC gain of the system with transfer function

$$H(s) = \frac{s^2 + 5s + 3}{s^3 + 4s + 2s + 5}$$

All poles of $H(s)$ are strictly stable (we will see this later using the *Routh–Hurwitz criterion*), so

$$y(\infty) = H(s)\Big|_{s=0} = \frac{3}{5}.$$