Plan of the Lecture

- **Review:** stability; Routh–Hurwitz criterion
- **Today’s topic:** basic properties and benefits of feedback control

Goal:
understand the difference between open-loop and closed-loop (feedback) control; examine the benefits of feedback:
reference tracking and disturbance rejection; reduction of sensitivity to parameter variations; improvement of time response.

Reading:
FPE, Section 4.1; lab manual
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**Reading:** FPE, Section 4.1; lab manual
Two Basic Control Architectures

- Open-loop control

\[ R \rightarrow K \rightarrow U \rightarrow P \rightarrow Y \]

- Feedback (closed-loop) control

\[ R \rightarrow \text{error} \rightarrow E \rightarrow K \rightarrow U \rightarrow P \rightarrow Y \]

Here, \( W \) is a \textit{disturbance}; \( K \) is \textit{not necessarily} a static gain
Basic Objectives of Control

- track a given reference
- reject disturbances
- meet performance specs

Intuitively, the difference between the open-loop and the closed-loop architectures is clear (think cruise control ...)

![Diagram of control systems](image)
Open-Loop Control

- Cheaper/easier to implement (no sensor required)
- Does not destabilize the system

\[ W = KP \]

\{ \text{poles of } KP \} = \{ \text{poles of } K \} \cup \{ \text{poles of } P \} \]
Open-Loop Control

- cheaper/easier to implement (no sensor required)

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Open-Loop Control

▶ cheaper/easier to implement (no sensor required)
▶ does not destabilize the system

e.g., if both $K$ and $P$ are stable (all poles in OLHP),

$$\frac{Y}{R} = KP$$

is also stable:
Open-Loop Control

- cheaper/easier to implement (no sensor required)
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E.g., if both $K$ and $P$ are stable (all poles in OLHP),

$$\frac{Y}{R} = KP$$

is also stable:

$$\{\text{poles of } KP\} = \{\text{poles of } K\} \cup \{\text{poles of } P\}$$
Feedback Control

$W$

$R$

$E$

$K$

$U$

$P$

$Y$

- More difficult/expensive to implement (requires a sensor and an information path from controller to actuator).
- May destabilize the system: $Y = KP + 1$ has new poles, which may be unstable.
- But: feedback control is the only way to stabilize an unstable plant (this was the Wright brothers' key insight).
Feedback Control

- more difficult/expensive to implement (requires a sensor and an information path from controller to actuator)

\[ R \xrightarrow{+} E \xrightarrow{K} U \xrightarrow{P} Y \]

\[ W \]

\[ Y = KP_1 + KP \]

but: feedback control is the only way to stabilize an unstable plant (this was the Wright brothers' key insight)
Feedback Control

- more difficult/expensive to implement (requires a sensor and an information path from controller to actuator)
- may destabilize the system:

\[
\frac{Y}{R} = \frac{KP}{1 + KP}
\]

has new poles, which may be unstable
Feedback Control

- more difficult/expensive to implement (requires a sensor and an information path from controller to actuator)
- may destabilize the system:

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\frac{Y}{R} = \frac{KP}{1 + KP}
\]

has new poles, which may be unstable

- but: feedback control is the *only way* to stabilize an unstable plant (this was the Wright brothers’ key insight)
Benefits of Feedback Control

Feedback control:
- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

![Feedback control diagram]

Feedback control:
Benefits of Feedback Control

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Feedback control:
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Case Study: DC Motor

Inputs: \( v_a \) – input voltage
\( \tau_e \) – load/disturbance torque

Outputs: \( \omega_m \) – angular speed of the motor
Case Study: DC Motor

Inputs: \( v_a \) – input voltage
\( \tau_e \) – load/disturbance torque

Outputs: \( \omega_m \) – angular speed of the motor

Transfer function:

\[
\Omega_m = \frac{A}{\tau s + 1} V_a + \frac{B}{\tau s + 1} T_e
\]

\( \tau \) – time constant

\( A, B \) – system gains

Objective: have \( \Omega_m \) approach and track a given reference \( \Omega_{\text{ref}} \) in spite of disturbance \( T_e \).
Case Study: DC Motor

Inputs: $v_a$ – input voltage
$\tau_e$ – load/disturbance torque

Outputs: $\omega_m$ – angular speed of the motor

Transfer function:

$$\Omega_m = \frac{A}{\tau s + 1} V_a + \frac{B}{\tau s + 1} T_e$$

$\tau$ – time constant
$A, B$ – system gains

Objective: have $\Omega_m$ approach and track a given reference $\Omega_{ref}$ in spite of disturbance $T_e$. 

Diagram:

- $T_e$ flows into $B/A$
- $B/A$ feeds into $A/(\tau s + 1)$
- $V_a$ is added to $A/(\tau s + 1)$
- Resulting signal flows into $\Omega_m$
Case Study: DC Motor

Inputs:
- $v_a$ – input voltage
- $\tau_e$ – load/disturbance torque

Outputs:
- $\omega_m$ – angular speed of the motor

Transfer function:

$$\Omega_m = \frac{A}{\tau s + 1} v_a + \frac{B}{\tau s + 1} T_e$$

$\tau$ – time constant
$A, B$ – system gains

Objective: have $\Omega_m$ approach and track a given reference $\Omega_{\text{ref}}$ in spite of disturbance $T_e$. 
Two Control Configurations

- **Open-loop control**

- **Feedback (closed-loop) control**
Disturbance Rejection

**Goal:** maintain $\omega_m = \omega_{\text{ref}}$ in steady state in the presence of constant disturbance.
Disturbance Rejection

Goal: maintain $\omega_m = \omega_{\text{ref}}$ in steady state in the presence of constant disturbance.

Open-loop:
Disturbance Rejection

**Goal:** maintain $\omega_m = \omega_{\text{ref}}$ in steady state in the presence of *constant* disturbance.

Open-loop:

- the controller receives *no information* about the disturbance $\tau_e$ (the only input is $\omega_{\text{ref}}$, no feedback signal from anywhere else)
Disturbance Rejection

Goal: maintain $\omega_m = \omega_{\text{ref}}$ in steady state in the presence of constant disturbance.

Open-loop:

- the controller receives *no information* about the disturbance $\tau_e$ (the only input is $\omega_{\text{ref}}$, no feedback signal from anywhere else)
- so, let’s attempt the following: design for *no disturbance* (i.e., $\tau_e = 0$), then see how the system works in general
First assume zero disturbance:

\[ \Omega_{\text{ref}} \xrightarrow{K_{\text{ol}}} V_a \xrightarrow{\frac{A}{\tau s + 1}} \Omega_m \]

Transfer function:

\[ A \tau s + 1 \] 

stable pole at \( s = -1/\tau \)

We want DC gain = 1

\[ \Omega_m = \frac{A}{\tau s + 1} \]

\[ V_a = K_{\text{ol}} \]

\[ \Omega_{\text{ref}} = A \cdot \Omega_{\text{ref}}(\text{for } T = 0) \]

What happens in the presence of nonzero \( T \)?

\[ \Omega_m = \frac{1}{A} \]

\[ \text{DC gain} = \frac{B}{A} \]

\[ T \text{e} \]

\[ \Omega_m(\infty) = \Omega_{\text{ref}} + B \tau \text{e} \]

\[ \text{step input} \]

\[ \text{step input} \]
Disturbance Rejection: Open-Loop Control

First assume zero disturbance:

\[
\Omega_{\text{ref}} \rightarrow K_{\text{ol}} \rightarrow V_a \rightarrow \frac{A}{\tau s + 1} \rightarrow \Omega_{\text{m}}
\]

Transfer function:

\[
\frac{A}{\tau s + 1} \quad (\text{stable pole at } s = -1/\tau)
\]
Disturbance Rejection: Open-Loop Control

First assume zero disturbance:

\[ \Omega_{\text{ref}} \rightarrow K_{\text{ol}} \rightarrow V_a \rightarrow \frac{A}{\tau s + 1} \rightarrow \Omega_m \]

Transfer function:

\[ \frac{A}{\tau s + 1} \] (stable pole at \( s = -1/\tau \))

We want DC gain = 1
Disturbance Rejection: Open-Loop Control

First assume zero disturbance:

\[
\begin{align*}
\Omega_{\text{ref}} & \rightarrow K_{\text{ol}} \rightarrow V_a \rightarrow \frac{A}{\tau s + 1} \rightarrow \Omega_m \\
\text{open-loop controller} & \quad \text{motor}
\end{align*}
\]

Transfer function:

\[
\frac{A}{\tau s + 1} \quad \text{(stable pole at } s = -1/\tau)\]

We want DC gain = 1

\[
\Omega_m = \frac{A}{\tau s + 1} V_a = \frac{K_{\text{ol}} A}{\tau s + 1} \Omega_{\text{ref}}
\]
Disturbance Rejection: Open-Loop Control

First assume zero disturbance:

\[ \Omega_{\text{ref}} \rightarrow K_{\text{ol}} \rightarrow \frac{A}{\tau s + 1} \rightarrow \Omega_{\text{m}} \]

Transfer function:

\[ \frac{A}{\tau s + 1} \] (stable pole at \( s = -1/\tau \))

We want DC gain = 1

\[ \Omega_{\text{m}} = \frac{A}{\tau s + 1} V_a = \frac{K_{\text{ol}} A}{\tau s + 1} \Omega_{\text{ref}} \]

Let’s just use constant gain: \( K_{\text{ol}} = 1/A \)
Disturbance Rejection: Open-Loop Control

First assume zero disturbance:

\[ \Omega_{\text{ref}} \xrightarrow{K_{\text{ol}}} V_a \xrightarrow{\frac{A}{\tau s + 1}} \Omega_m \]

Open-loop controller

Motor

Transfer function:

\[ \frac{A}{\tau s + 1} \] (stable pole at \( s = -1/\tau \))

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\[ \Omega_m = \frac{A}{\tau s + 1} V_a = \frac{K_{\text{ol}} A}{\tau s + 1} \Omega_{\text{ref}} \]

Let's just use constant gain: \( K_{\text{ol}} = 1/A \)

\[ \omega_m(\infty) = \frac{1}{A} \cdot A \cdot \omega_{\text{ref}} = \omega_{\text{ref}} \quad \text{(for } T_e = 0) \]
Disturbance Rejection: Open-Loop Control

First assume zero disturbance:

\[ \Omega_{\text{ref}} \xrightarrow{K_{\text{ol}}} V_a \xrightarrow{\frac{A}{\tau s + 1}} \Omega_m \]

Transfer function:

\[ \frac{A}{\tau s + 1} \quad \text{(stable pole at } s = -1/\tau) \]

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\[ \omega_m(\infty) = \frac{1}{A} \cdot A \cdot \omega_{\text{ref}} = \omega_{\text{ref}} \quad \text{(for } T_e = 0) \]

What happens in the presence of nonzero \( T_e \)?
Disturbance Rejection: Open-Loop Control

First assume zero disturbance:

\[ \Omega_{\text{ref}} \xrightarrow{K_{\text{ol}}} V_a \xrightarrow{A/\tau s + 1} \Omega_m \]

- **Open-loop controller**
- **Motor**

Transfer function:

\[ \frac{A}{\tau s + 1} \]  
(stable pole at \( s = -1/\tau \))

We want DC gain = 1

\[ \Omega_m = \frac{A}{\tau s + 1} V_a = \frac{K_{\text{ol}} A}{\tau s + 1} \Omega_{\text{ref}} \]

Let’s just use constant gain: \( K_{\text{ol}} = 1/A \)

\[ \omega_m(\infty) = \frac{1}{A} \cdot A \cdot \omega_{\text{ref}} = \omega_{\text{ref}} \quad \text{(for } T_e = 0) \]

What happens in the presence of nonzero \( T_e \)?

\[ \Omega_m = \underbrace{\frac{A}{\tau s + 1}}_{\text{DC gain}=1} \underbrace{\frac{1}{A} \Omega_{\text{ref}}}_{\text{DC gain}=1} + \underbrace{\frac{B}{\tau s + 1}}_{\text{DC gain}=B} T_e \]
Disturbance Rejection: Open-Loop Control

First assume zero disturbance:

\[ \Omega_{\text{ref}} \rightarrow K_{\text{ol}} \rightarrow V_a \rightarrow \frac{A}{\tau s + 1} \rightarrow \Omega_m \]

Transfer function:

\[ \frac{A}{\tau s + 1} \]  
(stable pole at \( s = -1/\tau \))

We want DC gain = 1

\[ \Omega_m = \frac{A}{\tau s + 1} V_a = \frac{K_{\text{ol}} A}{\tau s + 1} \Omega_{\text{ref}} \]

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\[ \omega_m(\infty) = \frac{1}{A} \cdot A \cdot \omega_{\text{ref}} = \omega_{\text{ref}} \quad \text{(for } T_e = 0) \]

What happens in the presence of nonzero \( T_e \)?

\[ \Omega_m = \frac{A}{\tau s + 1} \left( \frac{1}{A} \Omega_{\text{ref}} + \frac{B}{\tau s + 1} T_e \right) \]

\[ \Rightarrow \omega_m(\infty) = \omega_{\text{ref}} + B \frac{T_e}{\tau_e} \quad \text{step input} \]

\[ \Rightarrow \omega_m(\infty) = \omega_{\text{ref}} + B \tau_e \quad \text{step input} \]
Disturbance Rejection: Open-Loop Control

Steady-state motor speed for constant reference and constant disturbance:

\[ \omega_m(\infty) = \omega_{\text{ref}} + B\tau_e \]

Conclusion: in the absence of disturbances, reference tracking is good, but disturbance rejection is pretty poor. Steady-state error is determined by \( B \), and we have no control over it (and, in fact, cannot change this through any choice of controller \( K_{\text{ol}} \), no matter how clever).
Disturbance Rejection: Feedback Control

\[ V_a = K_{cl} E \]
\[ \Omega_m = \frac{A}{\tau s + 1} \]

\[ \Omega_m = \frac{A}{\tau s + 1} \]

\[ T_e \]

\[ B/A \]

\[ \Omega_{\text{ref}} \]

\[ K_{cl} \]

\[ \text{closed-loop controller} \]

\[ \text{motor} \]

\[ \text{error} \]
Disturbance Rejection: Feedback Control

\[ V_a = K_{cl} E \]
Disturbance Rejection: Feedback Control

\[ V_a = K_{cl} E = K_{cl} (\Omega_{\text{ref}} - \Omega_{m}) \]
Disturbance Rejection: Feedback Control

\[ V_a = K_{cl} E = K_{cl} (\Omega_{\text{ref}} - \Omega_m) \]

\[ \Omega_m = \frac{A}{\tau s + 1} K_{cl} (\Omega_{\text{ref}} - \Omega_m) + \frac{B}{\tau s + 1} T_e \]
Disturbance Rejection: Feedback Control

\[ V_a = K_{cl} E = K_{cl} (\Omega_{\text{ref}} - \Omega_m) \]

\[ \Omega_m = \frac{A}{\tau s + 1} K_{cl} (\Omega_{\text{ref}} - \Omega_m) + \frac{B}{\tau s + 1} T_e \]

Solve for \( \Omega_m \):
Disturbance Rejection: Feedback Control

\[ V_a = K_{cl} E = K_{cl} (\Omega_{\text{ref}} - \Omega_m) \]

\[ \Omega_m = \frac{A}{\tau s + 1} K_{cl} (\Omega_{\text{ref}} - \Omega_m) + \frac{B}{\tau s + 1} T_e \]

Solve for \( \Omega_m \): \((\tau s + 1)\Omega_m = AK_{cl} (\Omega_{\text{ref}} - \Omega_m) + B T_e\)
Disturbance Rejection: Feedback Control

\[ V_a = K_{cl} E = K_{cl} (\Omega_{\text{ref}} - \Omega_m) \]

\[ \Omega_m = \frac{A}{\tau s + 1} K_{cl} (\Omega_{\text{ref}} - \Omega_m) + \frac{B}{\tau s + 1} T_e \]

Solve for \( \Omega_m \):

\[ (\tau s + 1)\Omega_m = AK_{cl} (\Omega_{\text{ref}} - \Omega_m) + BT_e \]

\[ (\tau s + 1 + AK_{cl})\Omega_m = AK_{cl} \Omega_{\text{ref}} + BT_e \]
Disturbance Rejection: Feedback Control

\[ V_a = K_{cl} E = K_{cl} (\Omega_{\text{ref}} - \Omega_m) \]

\[ \Omega_m = \frac{A}{\tau s + 1} K_{cl} (\Omega_{\text{ref}} - \Omega_m) + \frac{B}{\tau s + 1} T_e \]

Solve for \( \Omega_m \):

\[ (\tau s + 1)\Omega_m = AK_{cl} (\Omega_{\text{ref}} - \Omega_m) + BT_e \]

\[ (\tau s + 1 + AK_{cl})\Omega_m = AK_{cl}\Omega_{\text{ref}} + BT_e \]

\[ \Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}} \Omega_{\text{ref}} + \frac{B}{\tau s + 1 + AK_{cl}} T_e \]
Disturbance Rejection: Feedback Control

\[ \Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}} \Omega_{\text{ref}} + \frac{B}{\tau s + 1 + AK_{cl}} T_e \]

(provided all transfer functions are strictly stable)
Disturbance Rejection: Feedback Control

\[ \Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}} \Omega_{\text{ref}} + \frac{B}{\tau s + 1 + AK_{cl}} T_e \]

(provided all transfer functions are strictly stable)

Assuming that the reference \( \omega_{\text{ref}} \) and the disturbance \( \tau_e \) are constant, we apply FVT:

\[ \omega_m(\infty) = \frac{AK_{cl}}{1 + AK_{cl}} \omega_{\text{ref}} + \frac{B}{1 + AK_{cl}} \tau_e \]
Disturbance Rejection: Feedback Control

Steady-state speed for constant reference and disturbance:

\[ \omega_m(\infty) = \frac{AK_{cl}}{1 + AK_{cl}} \omega_{ref} + \frac{B}{1 + AK_{cl}} \tau_e \]

Conclusions:

- \( \frac{AK_{cl}}{1 + AK_{cl}} \neq 1 \), but can be brought arbitrarily close to 1 when \( K_{cl} \to \infty \). Thus, steady-state tracking is good with high gain, but never quite as good as in open-loop case.

- \( \frac{B}{1 + AK_{cl}} \) is small (arbitrarily close to 0) for large \( K_{cl} \). Thus, much better disturbance rejection than with open-loop control.
Sensitivity to Parameter Variations

Consider again our DC motor model, with no disturbance:

\[
A \frac{\tau s + 1}{\tau s + 1} + A \frac{\tau s + 1}{\tau s + 1}
\]

Bode's sensitivity concept: In the "nominal" situation, we have the motor with DC gain = \( A \), and the overall transfer function, either open- or closed-loop, has some other DC gain (call it \( T \)).

Now suppose that, due to modeling error, changes in operating conditions, etc., the motor gain changes:

\[ A \rightarrow A + \delta A \]

This will cause a perturbation in the overall DC gain:

\[ T \rightarrow T + \delta T \]

(from calculus, to 1st order, \( \delta T \approx \frac{dT}{dA} \delta A \))

\[
\begin{align*}
\Omega_{\text{ref}} & \rightarrow K_{\text{ol}} \rightarrow V_a \\
& \rightarrow \frac{A}{\tau s + 1} \rightarrow \Omega_{m}
\end{align*}
\]

\[
\begin{align*}
\Omega_{\text{ref}} & \rightarrow K_{\text{cl}} \rightarrow \frac{A}{\tau s + 1} \\
& \rightarrow \Omega_{m}
\end{align*}
\]
Sensitivity to Parameter Variations

Consider again our DC motor model, with no disturbance:

\[ A \frac{V_a}{s + 1} \]

Bode’s sensitivity concept: In the “nominal” situation, we have the motor with DC gain = \( A \), and the overall transfer function, either open- or closed-loop, has some other DC gain (call it \( T \)).
Sensitivity to Parameter Variations

Consider again our DC motor model, with no disturbance:

![Diagram of DC motor model with open-loop and closed-loop controller]

Bode's sensitivity concept: In the “nominal” situation, we have the motor with DC gain = $A$, and the overall transfer function, either open- or closed-loop, has some other DC gain (call it $T$). Now suppose that, due to modeling error, changes in operating conditions, etc., the motor gain changes:

$$A \rightarrow A + \delta A$$

where $\delta A$ is a small perturbation.
Sensitivity to Parameter Variations

Consider again our DC motor model, with no disturbance:

\[
\begin{align*}
\Omega_{\text{ref}} & \rightarrow K_{\text{ol}} \rightarrow V_a \rightarrow \frac{A}{\tau s + 1} \rightarrow \Omega_m \\
\Omega_{\text{ref}} & \rightarrow K_{\text{cl}} \rightarrow \frac{A}{\tau s + 1} \rightarrow \Omega_m
\end{align*}
\]

Bode’s sensitivity concept: In the “nominal” situation, we have the motor with DC gain = \(A\), and the overall transfer function, either open- or closed-loop, has some other DC gain (call it \(T\)). Now suppose that, due to modeling error, changes in operating conditions, etc., the motor gain changes:

\[A \rightarrow A + \delta A\]

This will cause a perturbation in the overall DC gain:

\[T \rightarrow T + \delta T\]  (from calculus, to 1st order, \(\delta T \approx \frac{dT}{dA} \delta A\))
Sensitivity to Parameter Variations

\[ A \rightarrow A + \delta A \quad \text{(small perturbation in system gain)} \]
\[ T \rightarrow T + \delta T \quad \text{(resultant perturbation in overall DC gain)} \]

Bode’s sensitivity:

\[ S \triangleq \frac{\delta T/T}{\delta A/A} \]

Hendrik Wade Bode
(1905–1982)
Sensitivity to Parameter Variations

\[ A \rightarrow A + \delta A \quad \text{(small perturbation in system gain)} \]
\[ T \rightarrow T + \delta T \quad \text{(resultant perturbation in overall DC gain)} \]

Bode’s sensitivity:

\[ S \triangleq \frac{\delta T / T}{\delta A / A} \]

\( S = \) relative error
\[ = \frac{\text{normalized (percentage) error in } T}{\text{normalized (percentage) error in } A} \]
Sensitivity to Parameter Variations

Let’s compute $S$ for our DC motor control example, both open- and closed-loop.
Sensitivity to Parameter Variations

Let’s compute $S$ for our DC motor control example, both open- and closed-loop.

Open-loop:
Sensitivity to Parameter Variations

Let’s compute $S$ for our DC motor control example, both open- and closed-loop.

Open-loop:

- nominal case
Sensitivity to Parameter Variations

Let’s compute $S$ for our DC motor control example, both open- and closed-loop.

Open-loop:

- nominal case $T_{ol} = K_{ol} A = \frac{1}{A} A = 1$
Let’s compute $S$ for our DC motor control example, both open- and closed-loop.

**Open-loop:**
- nominal case \( T_{ol} = K_{ol} A = \frac{1}{A} A = 1 \)
- perturbed case
Sensitivity to Parameter Variations

Let’s compute $S$ for our DC motor control example, both open- and closed-loop.

Open-loop:

- nominal case $T_{ol} = K_{ol} A = \frac{1}{A} A = 1$
- perturbed case $A \rightarrow A + \delta A$
Sensitivity to Parameter Variations

Let’s compute $S$ for our DC motor control example, both open- and closed-loop.

**Open-loop:**

- **nominal case**
  
  $T_{\text{ol}} = K_{\text{ol}}A = \frac{1}{A}A = 1$

- **perturbed case**

  $A \rightarrow A + \delta A$

  $T_{\text{ol}} \rightarrow K_{\text{ol}}(A + \delta A)$
Sensitivity to Parameter Variations

Let’s compute $S$ for our DC motor control example, both open- and closed-loop.

**Open-loop:**

- **nominal case** $T_{ol} = K_{ol}A = \frac{1}{A}A = 1$
- **perturbed case**

  $A \rightarrow A + \delta A$

  $T_{ol} \rightarrow K_{ol}(A + \delta A) = \frac{1}{A} (A + \delta A)$
Sensitivity to Parameter Variations

Let’s compute $S$ for our DC motor control example, both open- and closed-loop.

**Open-loop:**

- **nominal case** 
  
  $T_{ol} = K_{ol} A = \frac{1}{A} A = 1$

- **perturbed case**

  
  $A \longrightarrow A + \delta A$

  
  $T_{ol} \longrightarrow K_{ol}(A + \delta A) = \frac{1}{A} (A + \delta A) = 1 + \frac{\delta A}{A}$

  
  \[ \underbrace{\frac{1}{A}}_{\text{design choice}} + \underbrace{\frac{\delta A}{A}}_{\delta T_{ol}} \]
Sensitivity to Parameter Variations

Let’s compute $S$ for our DC motor control example, both open- and closed-loop.

Open-loop:

- **nominal case**
  \[ T_{\text{ol}} = K_{\text{ol}} A = \frac{1}{A} A = 1 \]

- **perturbed case**
  \[ A \rightarrow A + \delta A \]
  \[ T_{\text{ol}} \rightarrow K_{\text{ol}} (A + \delta A) = \frac{1}{A} (A + \delta A) = \left( \frac{1}{T_{\text{ol}}} \right) + \frac{\delta A}{A} \]

Sensitivity:

\[ S_{\text{ol}} = \frac{\delta T_{\text{ol}}/T_{\text{ol}}}{\delta A_{\text{ol}}/A_{\text{ol}}} = \frac{\delta A/A}{\delta A/A} = 1 \]
Sensitivity to Parameter Variations

Let’s compute $S$ for our DC motor control example, both open- and closed-loop.

Open-loop:

- **nominal case**
  
  $T_{ol} = K_{ol}A = \frac{1}{A}A = 1$

- **perturbed case**

  $A \rightarrow A + \delta A$

  $T_{ol} \rightarrow K_{ol}(A + \delta A) = \frac{1}{A} (A + \delta A) = \frac{1}{T_{ol}} + \frac{\delta A}{\delta T_{ol}}$

Sensitivity: $S_{ol} = \frac{\delta T_{ol}/T_{ol}}{\delta A_{ol}/A_{ol}} = \frac{\delta A/A}{\delta A/A} = 1$

For example, a 5% error in $A$ will cause a 5% error in $T_{ol}$. 
Sensitivity to Parameter Variations

Closed-loop:

$T_{cl} = A_{\text{nominal}} + \delta A$

$T_{cl} \rightarrow T_{cl} + \delta T$

How to compute this?

Taylor expansion:

$T(A + \delta A) = T(A) + \frac{dT}{dA}(A)\delta A + \text{higher-order terms}$

In our case:

$\frac{dT_{cl}}{dA} = K_{cl} + A_{cl}K_{cl} - A_{cl}^2K_{cl}^2(1 + A_{cl})^2$

$\delta T_{cl} = K_{cl}(1 + A_{cl})^2 \delta A$
Sensitivity to Parameter Variations

Closed-loop:

- nominal case

\[ T_{cl} \rightarrow T_{cl} + \delta T \]

\[ A \rightarrow A + \delta A \]

How to compute this?

Taylor expansion:

\[ T(A + \delta A) = T(A) + dT \frac{\partial T}{\partial A} \delta A + \text{higher-order terms} \]

In our case:

\[ \frac{\partial T_{cl}}{\partial A} = K_{cl} (1 + AK_{cl})^2 \]

\[ \delta T_{cl} = K_{cl} (1 + AK_{cl})^2 \delta A \]
Sensitivity to Parameter Variations

Closed-loop:

- **nominal case**
  \[ T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}} \]
Sensitivity to Parameter Variations

Closed-loop:

- nominal case  \( T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}} \)
- perturbed case
Sensitivity to Parameter Variations

Closed-loop:

- nominal case
  \[ T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}} \]
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  \[ A \rightarrow A + \delta A \]
Sensitivity to Parameter Variations

Closed-loop:

- nomial case  \[ T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}} \]

- perturbed case

\[ A \rightarrow A + \delta A \quad T_{cl} \rightarrow T_{cl} + \delta T_{cl} \]

how to compute this?

Taylor expansion:

\[ T(A + \delta A) = T(A) + \frac{dT}{dA}(A)\delta A + \text{higher-order terms} \]

In our case:

\[ \frac{dT_{cl}}{dA} = K_{cl} \left( 1 + AK_{cl} \right)^2 \]

\[ \delta T_{cl} = K_{cl} \left( 1 + AK_{cl} \right)^2 \delta A \]
Sensitivity to Parameter Variations

Closed-loop:

- nominal case \( T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}} \)
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A \rightarrow A + \delta A \quad T_{cl} \rightarrow T_{cl} + \delta T_{cl}
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In our case:
Sensitivity to Parameter Variations

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In our case:

\[ \frac{dT_{cl}}{dA} = \frac{K_{cl}}{1 + AK_{cl}} - \frac{AK_{cl}^2}{(1 + AK_{cl})^2} \]
Sensitivity to Parameter Variations

Closed-loop:

- **nominal case** \( T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}} \)
- **perturbed case**

\[
A \rightarrow A + \delta A \quad T_{cl} \rightarrow T_{cl} + \delta T_{cl}
\]

Taylor expansion:

\[
T(A + \delta A) = T(A) + \frac{dT}{dA}(A)\delta A + \text{higher-order terms}
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In our case:

\[
\frac{dT_{cl}}{dA} = \frac{K_{cl}}{1 + AK_{cl}} - \frac{AK_{cl}^2}{(1 + AK_{cl})^2} = \frac{K_{cl}}{(1 + AK_{cl})^2}
\]
Sensitivity to Parameter Variations

Closed-loop:

- **nominal case** \[ T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}} \]

- **perturbed case**

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\]

\[
\delta T_{cl} = \frac{K_{cl}}{(1 + AK_{cl})^2} \delta A
\]

how to compute this?
Sensitivity to Parameter Variations

From before:

\[ \delta T_{cl} = \frac{K_{cl}}{(1 + AK_{cl})^2} \delta A \]

\[ T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}} \]

With high-gain feedback, we get smaller relative error due to parameter variations in the plant model.
Sensitivity to Parameter Variations

From before:

\[
\delta T_{cl} = \frac{K_{cl}}{(1 + AK_{cl})^2} \delta A
\]

\[
T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}}
\]

Therefore

\[
\frac{\delta T_{cl}}{T_{cl}} = \frac{K_{cl}}{(1 + AK_{cl})^2} \delta A = \frac{1}{1 + AK_{cl}} \frac{\delta A}{A}
\]
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Therefore

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Sensitivity:

\[ S_{cl} = \frac{\delta T_{cl}/T_{cl}}{\delta A/A} = \frac{1}{1 + AK_{cl}} \]
Sensitivity to Parameter Variations

From before:

\[
\delta T_{cl} = \frac{K_{cl}}{(1 + AK_{cl})^2} \delta A
\]

\[
T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}}
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Therefore

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\frac{\delta T_{cl}}{T_{cl}} = \frac{K_{cl}}{(1 + AK_{cl})^2} \frac{\delta A}{AK_{cl}} = \frac{1}{1 + AK_{cl}} \frac{\delta A}{A}
\]

Sensitivity: \( S_{cl} = \frac{\delta T_{cl}/T_{cl}}{\delta A/A} = \frac{1}{1 + AK_{cl}} \quad (\ll 1 \text{ for large } K_{cl}) \)
Sensitivity to Parameter Variations

From before:

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With high-gain feedback, we get smaller relative error due to parameter variations in the plant model.
Time Response

We still assume no disturbance: $\tau_e = 0$. 

The open-loop transfer function is:

$$\Omega_m = AK_{\text{cl}} \tau_s + 1 \Omega_{\text{ref}}$$

The pole at $s = -\frac{1}{\tau}$ implies that the transient response is:

$$e^{-t/\tau}$$

Here, $\tau$ is the time constant: the time it takes the system response to decay to $1/e$ of its starting value.

In the open-loop case, larger time constant means faster convergence to steady state. This is not affected by the choice of $K_{\text{cl}}$ in any way!
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So far, we have focused on DC gain only (steady-state response). What about transient response?
Time Response

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Open-loop

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Pole at \( s = -\frac{1}{\tau} \) \( \implies \) transient response is \( e^{-t/\tau} \)
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Time Response

Closed-loop

\[
\begin{align*}
\Omega_{\text{ref}} & \rightarrow K_{\text{cl}} \rightarrow \frac{A}{\tau s + 1} \rightarrow \Omega_{m} \\
\Omega_{m} & = \frac{AK_{\text{cl}}}{\tau s + 1 + AK_{\text{cl}}} \Omega_{\text{ref}}
\end{align*}
\]
Time Response

Closed-loop

\[ \Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}} \Omega_{ref} \]

Closed-loop pole at \( s = -\frac{1}{\tau} (1 + AK_{cl}) \)
Time Response

Closed-loop

\[ \Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}} \Omega_{ref} \]

Closed-loop pole at \( s = -\frac{1}{\tau} (1 + AK_{cl}) \)

(the only way to move poles around is via feedback)
Time Response

Closed-loop

\[ \Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}} \Omega_{ref} \]

*Closed-loop* pole at \( s = -\frac{1}{\tau} (1 + AK_{cl}) \)

(the only way to move poles around is *via feedback*).

Now the transient response is \( e^{-\frac{1+AK_{cl}}{\tau} t} \), with

\[ \text{time constant} = \frac{\tau}{1 + AK_{cl}} \]
Time Response
Closed-loop

\[ \Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}} \Omega_{ref} \]

Closed-loop pole at \( s = -\frac{1}{\tau} (1 + AK_{cl}) \)
(the only way to move poles around is via feedback)

Now the transient response is \( e^{-\frac{1+AK_{cl}}{\tau}t} \), with

\[
\text{time constant} = \frac{\tau}{1 + AK_{cl}}
\]

— for large \( K_{cl} \), we have a much smaller time constant, i.e.,

*fast*er convergence to steady-state.
Summary

Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Word of caution: high-gain feedback only works well for 1st-order systems; for higher-order systems, it will typically cause underdamping and instability. Thus, we need a more sophisticated design than just static gain. We will take this up in the next lecture with Proportional-Integral-Derivative (PID) control.
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