Plan of the Lecture

- Review: state-space models of systems; linearization
- Today’s topic: linear systems and their dynamic response
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- **Today’s topic:** linear systems and their dynamic response

*Goal:* develop a methodology for characterizing the output of a given system for a given input.
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- **Review:** state-space models of systems; linearization
- **Today’s topic:** linear systems and their dynamic response

*Goal:* develop a methodology for characterizing the output of a given system for a given input.

*Reading:* FPE, Section 3.1, Appendix A.
State-Space Models

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

where:
- \(x(t) \in \mathbb{R}^n\) is the state at time \(t\)
- \(u(t) \in \mathbb{R}^m\) is the input at time \(t\)
- \(y(t) \in \mathbb{R}^p\) is the output at time \(t\)
- \(A \in \mathbb{R}^{n \times n}\) is the dynamics matrix
- \(B \in \mathbb{R}^{n \times m}\) is the control matrix
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How do we determine the output \(y\) for a given input \(u\)?

Reminder: we will only consider single-input, single-output (SISO) systems, i.e., \(u(t), y(t) \in \mathbb{R}\) for all times \(t\) of interest. (\(m = p = 1\))
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Impulse Response
(Review from ECE 210)

\[
\dot{x} = Ax + Bu \\
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\]

Unit impulse (or Dirac's $\delta$-function):
1. $\delta(t) = 0$ for all $t \neq 0$
2. $\int_{a}^{b} \delta(t) \, dt = 1$ for all $a > 0$

It is useful to think of $\delta(t)$ as a limit of impulses of unit area:

\[
\text{area} = \frac{1}{\varepsilon} \quad \text{as} \quad \varepsilon \to 0, \quad \text{the impulse gets taller} \quad (1/\varepsilon \to +\infty), \quad \text{but the area}\]

under its graph remains at 1
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zero initial condition: \( x(0) = 0 \)
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Consider the input

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The system is \textit{linear} and \textit{time-invariant} (LTI), with zero I.C.:
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The function \( h \) is the \textit{impulse response} of the system.
Impulse Response

\[ \dot{x} = Ax + Bu \]
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zero initial condition: \( x(0) = 0 \)

Questions to consider:
1. If we know \( h \), how can we find the system's response to other (arbitrary) inputs?
2. If we don't know \( h \), how can we determine it?

We will start with Question 1.
Impulse Response

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Question: If we know \( h \), how can we find the system’s response to other (arbitrary) inputs?

Recall the sifting property of the \( \delta \)-function: for any function \( f \) which is “well-behaved” at \( t = \tau \),

\[ \int_{-\infty}^{\infty} f(t)\delta(t - \tau)\,dt = f(\tau) \]
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— any *reasonably regular* function can be represented as an integral of impulses!!
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Question: If we know \( h \), how can we find the system’s response to other (arbitrary) inputs?

By the sifting property, for a general input \( u(t) \) we can write

\[ u(t) = \int_{-\infty}^{\infty} u(\tau) \delta(t - \tau) d\tau. \]
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Now we recall the \textit{superposition principle}:
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Now we recall the superposition principle: the response of a linear system to a sum (or integral) of inputs is the sum (or integral) of the individual responses to these inputs.
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\[ u(t) = \int_{-\infty}^{\infty} u(\tau) \delta(t - \tau) d\tau \quad \rightarrow \quad y(t) = \int_{-\infty}^{\infty} u(\tau) h(t - \tau) d\tau \]

— the integral that defines \( y(t) \) is a

\[ \delta(t - \tau) \]
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— the integral that defines \( y(t) \) is a convolution of \( u \) and \( h \).
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\[ \begin{align*}
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zero initial condition: \( x(0) = 0 \)

Q: Does this formula provide a practical way of computing the output \( y \) for a given input \( u \)?

A: Not directly (computing convolutions is not exactly pleasant), but we can use Laplace transforms.
Impulse Response

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**Conclusion so far:** for zero initial conditions, the output is the convolution of the input with the system impulse response:

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**Q:** Does this formula provide a *practical* way of computing the output \( y \) for a given input \( u \)?

**A:** Not directly (computing convolutions is not exactly pleasant), but ...we can use Laplace transforms.
Laplace Transforms and the Transfer Function

*Reminder*: the *two-sided* Laplace transform of a function $f(t)$ is

$$F(s) = \int_{-\infty}^{\infty} f(\tau)e^{-s\tau} d\tau, \quad s \in \mathbb{C}$$
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time domain \quad frequency domain

<table>
<thead>
<tr>
<th>$u(t)$</th>
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<tbody>
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convolution in time domain $\leftrightarrow$ multiplication in frequency domain

$$y(t) = h(t) \star u(t) \quad \leftrightarrow \quad Y(s) = H(s)U(s)$$
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convolution in time domain \( \longleftrightarrow \) multiplication in frequency domain

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The Laplace transform of the impulse response

\[
H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \, d\tau,
\]

is called the *transfer function* of the system.
Laplace Transforms and the Transfer Function

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Limits of integration:
Laplace Transforms and the Transfer Function

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Limits of integration:
- We only deal with causal systems
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**Laplace Transforms and the Transfer Function**

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**Limits of integration:**

- We only deal with *causal* systems — output at time \( t \) is not affected by inputs at future times \( t' > t \)
- If the system is causal, then \( h(t) = 0 \) for \( t < 0 \) — \( h(t) \) is the response at time \( t \) to a unit impulse at time 0
- We will take all other possible inputs (not just impulses) to be 0 for \( t < 0 \), and work with *one-sided* Laplace transforms:
Laplace Transforms and the Transfer Function

\[ Y(s) = H(s)U(s), \quad \text{where } H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \]

Limits of integration:

- We only deal with causal systems — output at time \( t \) is not affected by inputs at future times \( t' > t \).
- If the system is causal, then \( h(t) = 0 \) for \( t < 0 \) — \( h(t) \) is the response at time \( t \) to a unit impulse at time 0.
- We will take all other possible inputs (not just impulses) to be 0 for \( t < 0 \), and work with one-sided Laplace transforms:

\[ y(t) = \int_{0}^{\infty} u(\tau) h(t - \tau) d\tau \]

\[ H(s) = \int_{0}^{\infty} h(\tau) e^{-s\tau} d\tau \]
Laplace Transforms and the Transfer Function

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Given \( u(t) \), we can find \( U(s) \) using tables of Laplace transforms or MATLAB.
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- Suppose we have a state-space model:

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\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
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Laplace Transforms and the Transfer Function

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- Suppose we have a state-space model:

\[
\begin{align*}
u & \rightarrow \dot{x} = Ax + Bu \\
y &= Cx
\end{align*}
\]

In this case, we have an exact formula:

\[
H(s) = C(I - As)^{-1}B \quad \text{(matrix inversion)}
\]

\[
h(t) = Ce^{At}B, \quad t \geq 0^- \quad \text{(matrix exponential)}
\]

— will not encounter this until much later in the semester.
Laplace Transforms and the Transfer Function

\[ Y(s) = H(s)U(s), \quad \text{where } H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau} \, d\tau \]
Laplace Transforms and the Transfer Function

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- So, how should we compute \( H(s) \) in practice?
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Try injecting some specific inputs and see what happens at the output.
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Let’s try \( u(t) = e^{st}, t \geq 0 \) \( (s \) is some fixed number)
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= e^{st} \int_{0}^{\infty} h(\tau)e^{-s\tau}d\tau \\
= e^{st}H(s)
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Laplace Transforms and the Transfer Function

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\[ = \int_{0}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \]

\[ = e^{st} \int_{0}^{\infty} h(\tau)e^{-s\tau}d\tau \]

\[ = e^{st}H(s) \]

– so, \( u(t) = e^{st} \) is multiplied by \( H(s) \) to give the output.
Example

\[
\dot{y} = -ay + u \quad \text{(think } y = x, \text{ full measurement)}
\]

\[
u(t) = e^{st} \quad \text{(always assume } u(t) = 0 \text{ for } t < 0)\]
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\[ \dot{y} = -ay + u \]  
(think \( y = x \), full measurement)

\[ u(t) = e^{st} \]  
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Let’s use the system model:

\[ \dot{y}(t) = \frac{d}{dt} \left( H(s) e^{st} \right) = sH(s)e^{st} \]
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Substitute into \( \dot{y} = -ay + u \):

\[ sH(s)e^{st} = -aH(s)e^{st} + e^{st} \quad (\forall s; t > 0) \]
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(\( \forall s; t > 0 \))

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Example

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\[ H(s) = \frac{1}{s + a} \]  \( \Rightarrow \)  
\[ y(t) = \frac{e^{st}}{s + a} \]
Example (continued)

\[ \dot{y} = -ay + u \]

\[ H(s) = \frac{1}{s + a} \]
Example (continued)

\[ \ddot{y} = -ay + u \]

\[ H(s) = \frac{1}{s + a} \]

Now we can fund the impulse response \( h(t) \) by taking the inverse Laplace transform — from tables,

\[ h(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases} \]
Determining the Impulse Response

\[ u(t) = e^{st}, \quad t \geq 0 \]
\[ x(0) = 0; \text{LTI system} \]
\[ \implies y(t) = e^{st}H(s) \]

Back to our two questions:
1. If we know \( h \), how can we find \( y \) for a given \( u \)?
2. If we don't know \( h \), how can we determine it?

We have answered Question 1. Now let's turn to Question 2.

One idea: inject the input \( u(t) = e^{st} \), determine \( y(t) \), compute \( H(s) = \frac{y(t)}{u(t)} \); repeat for all \( s \) of interest.

Q: Is this a good idea?
Determining the Impulse Response

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\[ u \rightarrow h \rightarrow y \]

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Q: Is this likely to work in practice?
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**Q:** Is this likely to work *in practice*?

**A:** No — \( e^{st} \) blows up very quickly if \( s > 0 \), and decays to 0 very quickly if \( s < 0 \).
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So we need *sustained, bounded signals* as inputs.
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A: No — \( e^{st} \) blows up very quickly if \( s > 0 \), and decays to 0 very quickly if \( s < 0 \).

So we need sustained, bounded signals as inputs.

This is possible if we allow \( s \) to take on complex values.
Review: Complex Numbers

\[ s = a + jb \]

— rectangular form

Polar form:

\[ s = re^{j\phi} \]

\[ r = \sqrt{a^2 + b^2} \quad \text{(magnitude)} \]

\[ \phi = \angle s = \tan^{-1} \left( \frac{b}{a} \right) \quad \text{(phase)} \]

Euler's formula:

\[ e^{j\phi} = \cos \phi + j \sin \phi \]
Review: Complex Numbers

$s = a + j b$ — rectangular form

Polar form:

$s = re^{j\varphi}$

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Review: Complex Numbers

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(phase)

Euler’s formula:
\[ e^{j \varphi} = \cos \varphi + j \sin \varphi \]
Frequency Response

\[ u(t) = A \cos(\omega t) \quad A - \text{amplitude}; \ \omega - (\text{angular}) \ \text{frequency, rad/s} \]
Frequency Response

\[ u(t) = A \cos(\omega t) \quad A \text{ – amplitude; } \omega \text{ – (angular) frequency, rad/s} \]

From Euler’s formula:

\[ A \cos(\omega t) = \frac{A}{2} \left( e^{j\omega t} + e^{-j\omega t} \right) \]
Frequency Response

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By linearity, the response is
Frequency Response

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From Euler’s formula:

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By linearity, the response is

\[ y(t) = \frac{A}{2} \left( H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t} \right) \]

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Frequency Response

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complex conjugate
Frequency Response

\[
u(t) = A \cos(\omega t) \quad A \text{ – amplitude; } \omega \text{ – (angular) frequency, rad/s}
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where \( H(j\omega) = \int_{0}^{\infty} h(\tau)e^{-j\omega \tau} d\tau \)

\[
H(-j\omega) = \int_{0}^{\infty} h(\tau)e^{j\omega \tau} d\tau = \overline{H(-j\omega)}
\]

\( \text{complex conjugate} \)
Frequency Response

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where \( H(j\omega) = \int_0^\infty h(\tau)e^{-j\omega \tau} d\tau \)

\[ H(-j\omega) = \int_0^\infty h(\tau)e^{j\omega \tau} d\tau = \overline{H(-j\omega)} \]

(recall that \( h(\tau) \) is real-valued)
Frequency Response

\[ u(t) = A \cos(\omega t) \quad \longrightarrow \quad y(t) = \frac{A}{2} \left( H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t} \right) \]
**Frequency Response**

\[ u(t) = A \cos(\omega t) \quad \rightarrow \quad y(t) = \frac{A}{2} \left( H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t} \right) \]

\[ H(j\omega) \in \mathbb{C} \quad \implies \quad H(j\omega) = M(\omega)e^{j\phi(\omega)} \]
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Frequency Response

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Therefore,

\[ y(t) = \frac{A}{2} M(\omega) \left[ e^{j(\omega t + \varphi(\omega))} + e^{-j(\omega t + \varphi(\omega))} \right] \]
Frequency Response

\[ \begin{align*}
  u(t) &= A \cos(\omega t) \\
  \implies y(t) &= \frac{A}{2} \left( H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t} \right)
\end{align*} \]

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Therefore,

\[ y(t) = \frac{A}{2} M(\omega) \left[ e^{j(\omega t + \varphi(\omega))} + e^{-j(\omega t + \varphi(\omega))} \right] \]
\[ = AM(\omega) \cos (\omega t + \varphi(\omega)) \]
Frequency Response

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  u(t) &= A \cos(\omega t) \quad \longrightarrow \quad y(t) = \frac{A}{2} \left( H(j\omega) e^{j\omega t} + H(-j\omega) e^{-j\omega t} \right) \\
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  \text{Therefore,} \\
  y(t) &= \frac{A}{2} M(\omega) \left[ e^{j(\omega t + \varphi(\omega))} + e^{-j(\omega t + \varphi(\omega))} \right] \\
  &= AM(\omega) \cos(\omega t + \varphi(\omega)) \quad \text{(only true in steady state)}
\end{align*}
\]
Frequency Response

\begin{align*}
  u(t) &= A \cos(\omega t) \quad \longrightarrow \quad y(t) = \frac{A}{2} \left( H(j\omega) e^{j\omega t} + H(-j\omega) e^{-j\omega t} \right) \\
  H(j\omega) \in \mathbb{C} \quad \Longrightarrow \quad H(j\omega) &= M(\omega) e^{j\varphi(\omega)} \\
  H(-j\omega) &= M(\omega) e^{-j\varphi(\omega)}
\end{align*}

Therefore,

\[ y(t) = \frac{A}{2} M(\omega) \left[ e^{j(\omega t + \varphi(\omega))} + e^{-j(\omega t + \varphi(\omega))} \right] = AM(\omega) \cos(\omega t + \varphi(\omega)) \]  

(only true in steady state)

The (steady-state) response to a cosine signal with amplitude \( A \) and frequency \( \omega \) is still a cosine signal with amplitude \( AM(\omega) \), same frequency \( \omega \), and phase shift \( \varphi(\omega) \)
Frequency Response

\[ u(t) = A \cos(\omega t) \quad \rightarrow \quad y(t) = A \begin{pmatrix} M(\omega) \\ \cos(\omega t + \phi(\omega)) \end{pmatrix} \]

- amplitude magnification
- phase shift

Still an incomplete picture:

▶ What about response to general signals (not necessarily sinusoids)? — always given by

\[ Y(s) = H(s) U(s) \]

▶ What about response under nonzero I.C.'s? — we will see that, if the system is stable, then total response = transient response (depends on I.C.) + steady-state response (independent of I.C.) — need more on Laplace transforms
Frequency Response

\[ u(t) = A \cos(\omega t) \quad \rightarrow \quad y(t) = A M(\omega) \cos(\omega t + \varphi(\omega)) \]

Still an incomplete picture:
Frequency Response

\[ u(t) = A \cos(\omega t) \quad \rightarrow \quad y(t) = A \cdot M(\omega) \cos(\omega t + \varphi(\omega)) \]

Still an incomplete picture:

- What about response to general signals (not necessarily sinusoids)?
Frequency Response

\[ u(t) = A \cos(\omega t) \quad \rightarrow \quad y(t) = A \left[ M(\omega) \cos(\omega t + \varphi(\omega)) \right] \]

Still an incomplete picture:

- What about response to general signals (not necessarily sinusoids)? — always given by \[ Y(s) = H(s)U(s) \]
Frequency Response

\[ u(t) = A \cos(\omega t) \quad \rightarrow \quad y(t) = A \left( M(\omega) \cos(\omega t + \varphi(\omega)) \right) \]

Still an incomplete picture:

- What about response to general signals (not necessarily sinusoids)? — always given by \( Y(s) = H(s)U(s) \)
- What about response under nonzero I.C.’s?
Frequency Response

\[ u(t) = A \cos(\omega t) \quad \rightarrow \quad y(t) = A(M(\omega)) \cos(\omega t + \varphi(\omega)) \]

Still an incomplete picture:

- What about response to general signals (not necessarily sinusoids)? — always given by \( Y(s) = H(s)U(s) \)
- What about response under nonzero I.C.’s?— we will see that, if the system is stable, then

\[
\text{total response} = \text{transient response} \quad + \quad \text{steady-state response}
\]

(\text{depends on I.C.}) + (\text{independent of I.C.})
Frequency Response

\[ u(t) = A \cos(\omega t) \quad \rightarrow \quad y(t) = A \ M(\omega) \ \cos \left( \omega t + \varphi(\omega) \right) \]

Still an incomplete picture:

- What about response to general signals (not necessarily sinusoids)? — always given by \( Y(s) = H(s)U(s) \)
- What about response under *nonzero I.C.’s*?— we will see that, if *the system is stable*, then

\[
\text{total response} = \ \text{transient response (depends on I.C.)} \ + \ \text{steady-state response (independent of I.C.)}
\]

— need more on Laplace transforms