ECE 486: A note on sensitivity

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Fall 2015, UIUC

1. We briefly discuss Bode's sensitivity in class. Sensitivity aims to quantify how sensitive the *behavior* of a controlled system is to errors in modeling, implementation, etc. In other words, it aims to address the following scenario: Suppose you are given a parametric model of a system and you design a controller that meets some specifications. If there are small perturbations to the parameters, which your controller is designed for, how far off the controlled system is from the specifications. Of course, we would want the sensitivity to be as small as possible.

   In general, and as we will see below, we also allow variations in the parameters of the *controller*. Since the controller has to be physically implemented anyway, thus it is subject to the same potential variations due to errors in modelling (e.g. a resistor does not meet its specified value exactly, etc.)

2. To be more specific, suppose you are asked to compute the sensitivity of a *target specification*, T, (e.g. DC gain, rise-time) as a function of a parameter, A_1. A priori, the target T depends on all parameters $A_1, \ldots, A_p$ entering the model of the system *and* the controller. So we can write T as a function of these parameters

   $$T(A_1, \ldots, A_p).$$

   The sensitivity of T with respect to A is defined as

   $$S := \frac{\frac{\delta T}{T}}{\frac{\delta A_1}{A_1}},$$

   which is the normalized (that is, divided by T) variation δT in the target T over the normalized variation in the parameter A_1.

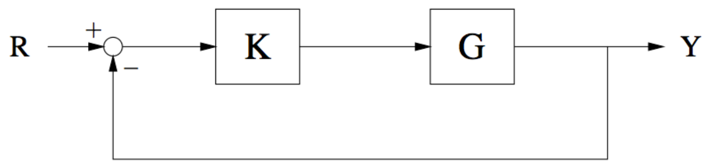
3. From calculus, we know that

   $$\delta T \simeq \frac{dT}{dA_1} \delta A_1.$$

   Hence the sensitivity of T with respect to A_1 is

   $$S = \frac{dT}{dA_1} \frac{A_1}{T}.$$

   Notice how you can get to the above formula directly from our original definition of sensitivity by replacing δ with d.



4. Let us look at an example. Consider the first order system with transfer function $G(s) = \frac{A}{\tau s+1}$ with constant feedback K in a unity feedback scheme:

   The closed-loop transfer function is then

$$H_{cl}(s) = \frac{Y}{R} = \frac{AK}{\tau s + 1 + AK}$$

   Assume that you design your control K such that the *rise-time* meets some specification. Suppose that an optimal value of K, denoted as K^*, for the feedback gain. What is the sensitivity of the **rise-time with respect to A?**

5. The target here is the rise-time, and the parameters are A, K and τ. Thus, we need to evaluate $\frac{dT(A,K,\tau)}{dA}$, for τ and K fixed.

   Recall that for a first order system, we had the formula

$$T = \frac{2.2}{a}$$

   where $a > 0$ was the position of the pole of the transfer function (that is, the pole was at $s = -a$). We see that in our case, the pole of the closed-loop transfer function is at $s = -\frac{1+AK}{\tau}$. Therefore, we have

$$T = \frac{2.2\tau}{1+AK}.$$

   A straightforward calculation yields

$$\frac{dT}{dA} = -\frac{2.2\tau K}{(1+AK)^2}.$$

   Multiplying this equality by A/T, we obtain

$$S = -\frac{AK}{1+AK}.$$

   We can see that small values of the gain K yield a better (closer to zero) sensitivity, and in the worst case scenario ($K = \pm\infty$), the sensitivity is ± 1. In particular, the sensitivity of our design feedback K^* is given by $-AK^*/(1+AK^*)$. Also, note that a positive change in A implies a negative change in the rise-time.