ECE 486, Fall 2017 **PROBLEM SET 6 - SOLUTION** **Due: October 31, 2017**

Reading: FPE, Sections 6.1, 6.2, 6.4-6.6.

Problems: (you can use MATLAB in all problems, but you must explain all steps and justify all answers)

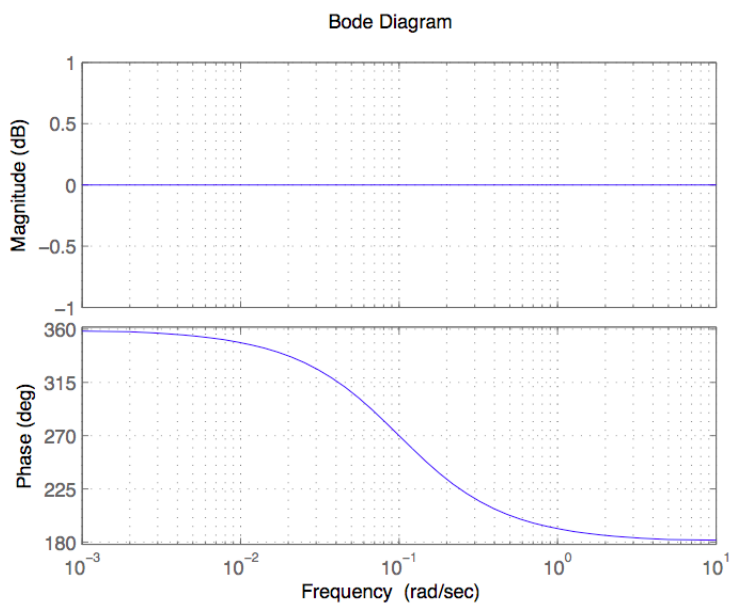
1. A delay of length T can be expressed as an LTI system with transfer function $G_T(s) = e^{-Ts}$. Using the approximation $e^x \approx 1 + x$, $x \sim 0$, we obtain the low-frequency approximation,

$$e^{-Ts} \approx \frac{1 - Ts/2}{1 + Ts/2}$$

Sketch *by hand* a Bode plot of this first-order approximation for e^{-Ts}, and on the same plot provide a sketch of the delay $G_T(j\omega)$ (note that the phase of $e^{-j\omega}$ is equal to $-\omega$). Indicate the range of ω for which the approximation accurate in terms of the delay T. What goes wrong at high frequencies?

Solution:

The figure below shows an example Bode plot when $T = 10$.



$$G_T(s) = e^{-Ts} \approx \frac{1 - Ts/2}{1 + Ts/2}$$

Substituting $s = jw$, we get

$$G_T(j\omega) = e^{-j\omega T} \approx \frac{1 - j\frac{T}{2}\omega}{1 + j\frac{T}{2}\omega}$$

The magnitude is

$$\begin{aligned} |G_T(j\omega)|_{dB} &= 20\log|G_T(j\omega)| = 20\log|e^{-j\omega T}| \\ &\approx 20\log|1 - j\frac{T}{2}\omega| - 20\log|1 + j\frac{T}{2}\omega| \end{aligned}$$

The phase is given by

$$\angle G_T(j\omega) = \angle e^{-j\omega T} = -\omega T \approx \angle(1 - j\frac{T}{2}\omega) - \angle(1 + j\frac{T}{2}\omega)$$

Note:
At $\omega = \dfrac{2}{T}$: $1 - j\dfrac{T}{2}\omega = 1 - j$, $1 + j\dfrac{T}{2}\omega = 1 + j$
At $\omega = \dfrac{0.2}{T}$: $1 - j\dfrac{T}{2}\omega = 1 - j0.1 \approx 1$, $1 + j\dfrac{T}{2}\omega = 1 + j0.1 \approx 1$
At $\omega = \dfrac{20}{T}$: $1 - j\dfrac{T}{2}\omega = 1 - j10 \approx -j10$, $1 + j\dfrac{T}{2}\omega = 1 + j \approx j10$

ω	$\lvert G_T(j\omega)\rvert$	$\angle G_T(j\omega)$ approx	$-\omega T$
$0.2/T$	$\dfrac{\lvert 1-j0.1\rvert}{\lvert 1+j0.1\rvert} \approx 1$	$0°$	$-11°$
$2/T$	$\dfrac{\lvert 1-j\rvert}{\lvert 1+j\rvert} = 1$	$-90°$	$-114°$
$20/T$	$\dfrac{\lvert 1-j10\rvert}{\lvert 1+j10\rvert} \approx 1$	$-180°$	$-65°$
$200/T$	$\dfrac{\lvert 1-j100\rvert}{\lvert 1+j100\rvert} \approx 1$	$-180°$	$60°$
$\cdots$	$\cdots$	$\cdots$	$\cdots$

From the table, we can see the range of ω for which the approximation accurate is $\omega \ll \frac{2}{T}$.

2. Show that for the transfer function $KG(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$, the phase margin is independent of ω_n and is given by

$$PM = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right).$$

Solution:

$$KG(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$$

To calculate the phase margin, we first find the gain-crossover-frequency (ω_c):

$$\begin{aligned}
\left|KG(j\omega)\right|\Big|_{\omega=\omega_c} = 1 &\Rightarrow \frac{\omega_n^2}{|-\omega_c^2 + 2j\omega_n\omega_c\zeta|} = 1 \\
&\Rightarrow \frac{\omega_n^2}{\sqrt{\omega_c^4 + 4\omega_n^2\omega_c^2\zeta^2}} = 1 \\
&\Rightarrow \omega_n^4 = \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2 \\
&\Rightarrow \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2 - \omega_n^4 = 0 \\
&\Rightarrow \begin{aligned} \omega_c^2 &= -2\zeta^2\omega_n^2 + \omega_n^2\sqrt{4\zeta^4+1} \\ &= \left(\sqrt{4\zeta^4+1} - 2\zeta^2\right)\omega_n^2 \end{aligned}
\end{aligned}$$

$$\begin{aligned}
KG(j\omega) &= \frac{\omega_n^2}{-\omega_c^2 + 2j\zeta\omega_n\omega_c} \\
\Rightarrow \quad \angle KG(j\omega) &= -\tan^{-1}\frac{2\zeta\omega_n\omega_c}{-\omega_c^2} \\
&= \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}
\end{aligned}$$

Note that $\theta = \tan^{-1} x \iff \pi + \theta = \tan^{-1}(x)$.

3. Consider the transfer function $G(s) = \dfrac{1}{s(s^2+4s+8)}$, which already appeared in Problem Sets 5 and 6.

a) Derive the value of K for which the closed-loop characteristic equation $1 + KG(s)$ has roots on the $j\omega$-axis.

b) For this value of K, make the Bode plot of $KG(s)$ using MATLAB and explain how you can confirm the presence of $j\omega$-axis closed-loop poles using this plot.

c) Compute the gain and phase margins for $K = 12$ using the corresponding Bode plot.

d) Determine the gain K that gives the phase margin of $60°$.

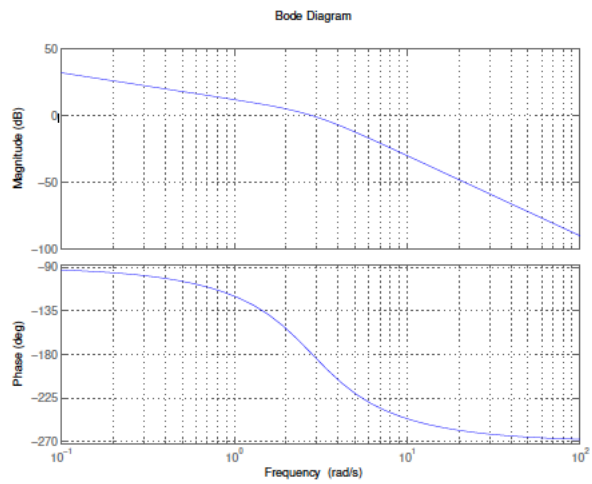
e) Plot the step responses of the closed-loop systems for $K = 12$ and the K you found in part d). Which system has better damping (smaller overshoot)? Why?

Solution:

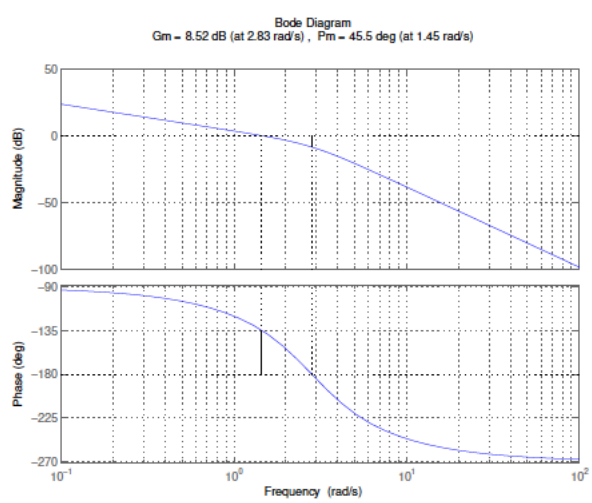
$$G(s) = \frac{1}{s(s^2 + 4s + 8)}$$

a) The critical value for K is 32, which causes roots of closed-loop system lie on $s = \pm j\sqrt{8} = \pm j2.8284$.

b) The attached Bode plot for $KG(s)$ ($K = 32$) shows that both of GM = PM = $0|_{\omega=\sqrt{8}}$, which shows that at ($\omega = \sqrt{8}$) the $|KG(j\omega)| = 1$ and $\angle KG(j\omega) = -\pi$ which are equivalent to gain and phase conditions of Root Locus.

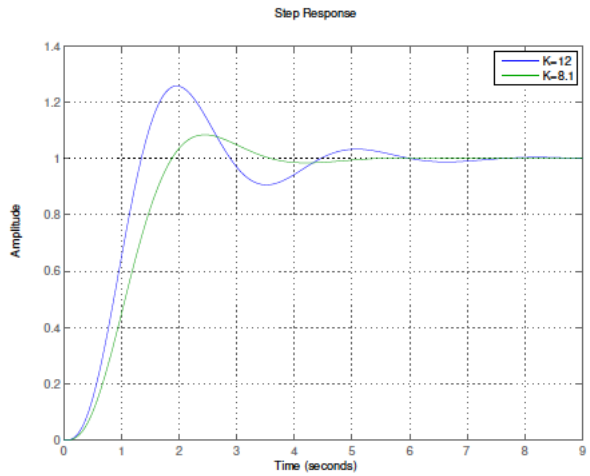


c) Attached plot shows GM = 8.52 dB (2.67) at $\omega = \sqrt{8}$ and PM = $45.5°$ at $\omega = 1.45$.



d) According to the phase plot, for PM = $60°$, we need $\omega_c = 1$, plugging this in $|KG(j\omega_c)| = 1$
$\therefore K = \sqrt{65} \approx 8.1$

e) System with $K = 8.1$, because larger PM is equivalent to larger ζ, and larger ζ is equivalent to smaller overshoot!



4. For the system

$$G(s) = \frac{10}{(\frac{s}{0.2}+1)(\frac{s}{0.5}+1)},$$

we want to design a lead/lag controller that provides bandwidth of at least 2, PM of at least 60°, and steady-state tracking of constant references within 1%.

a) For the controller derived in class:

$$KD(s) = 4\frac{\frac{s}{0.8}+1}{\frac{s}{5}+1} \cdot \frac{s+0.05}{s+0.02}$$

compute the PM, bandwidth, and steady-state tracking error to verify if the specs are met.

b) Suppose that in addition to the above specs, the bandwidth cannot exceed 6. Modify the design to incorporate this new spec, and verify that it indeed works.

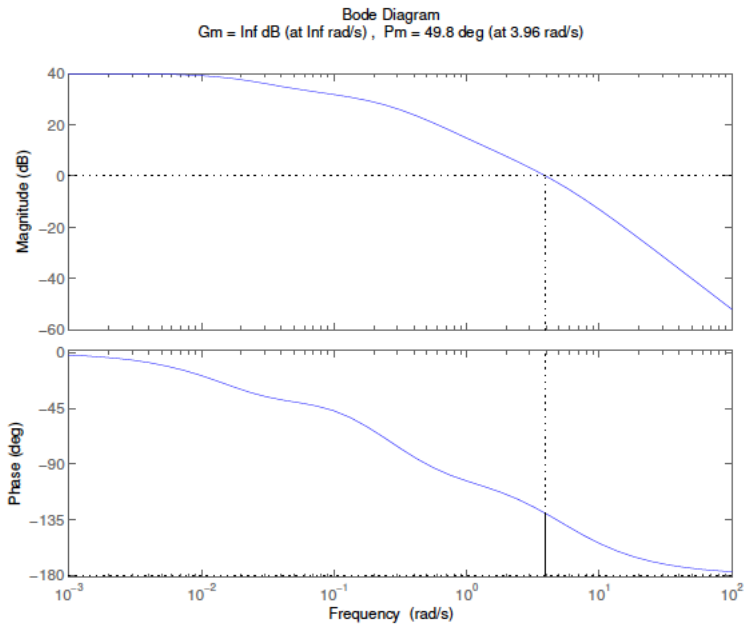
Solution:

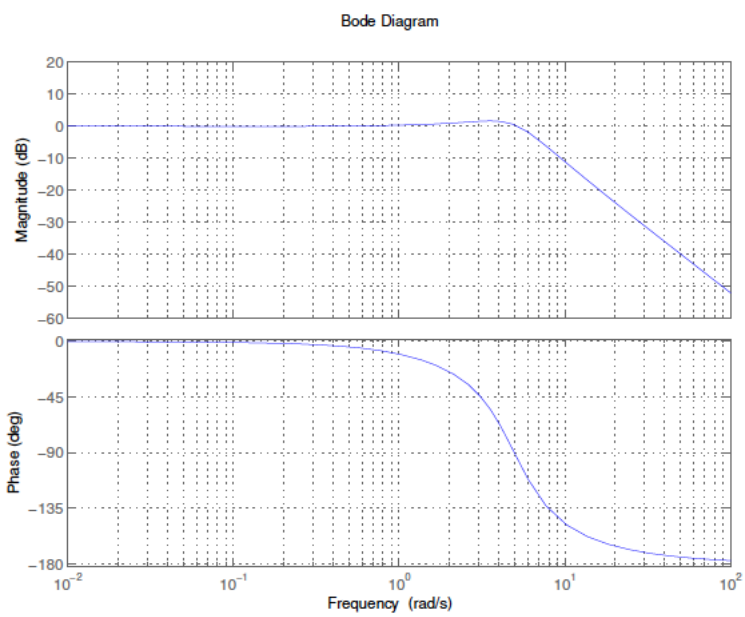
$$KD(s) = 4 \cdot \frac{\frac{s}{0.8}+1}{\frac{s}{5}+1} \cdot \frac{s+0.05}{s+0.02}.$$

(a) The attached Bode shows that PM $\approx 50°$ with $\omega_c = 4$ so the phase-margin is NOT achieved. The error is

$$e(\infty) = \frac{1}{1+KD(0)G(0)} = \frac{1}{101} < 0.01$$

and the closed-loop bode shows that BW ≈ 6.36.



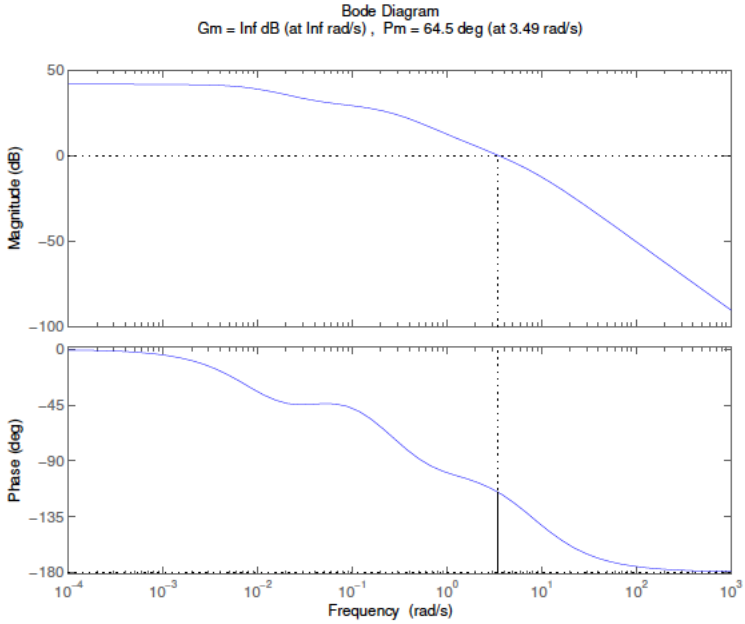


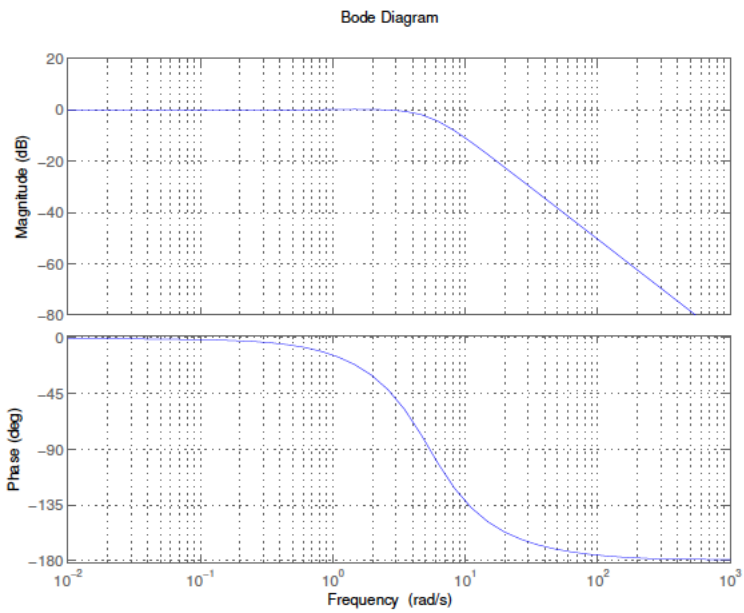
(b) To bring down BW (it's > 6), we need to reduce ω_c, also we need to increase our PM so we start with the lag controller from part 1(b) and pull the lead pole a little further to increase the phase which is provided by the lead controller and adjust the gain for error requirements:

$$KD(s) = 3 \cdot \frac{\frac{s}{0.8}+1}{\frac{s}{8}+1} \cdot \frac{s+0.04}{s+0.01}.$$

The attached Bodes show that PM $= 62°$ and BW $= 5.34$ and

$$e(\infty) = \left.\frac{1}{1+KD(s)G(s)}\right|_{s=0} = \frac{1}{121} < 0.01.$$


Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 64.5 deg (at 3.49 rad/s)
Magnitude (dB)
Phase (deg)
Frequency (rad/s)


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)