Plan of the Lecture

- **Review**: frequency-domain design method.
- **Today’s topic**: introduction to state-space design.

**Goal**: introduce basic notions of state-space control: different state-space realizations of the same transfer function; several canonical forms of state-space systems; controllability matrix.

**Reading**: FPE, Chapter 7
Frequency-Domain vs. State-Space

- 90% of industrial controllers are designed using frequency-domain methods (PID is a popular architecture)
- 90% of current research in systems and control is in the state-space framework

To be able to talk to control engineers and follow progress in the field, we need to know both methods and understand the connections between them.
The state-space approach reveals *internal system architecture* for a given transfer function.

The mathematics is different: heavy use of *linear algebra*.

This is just a short introduction; to learn this material properly, take ECE 515.
A General State-Space Model

\[ \begin{align*} 
\text{state } x &= \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \\
\text{input } u &= \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m \\
\text{output } y &= \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \in \mathbb{R}^p \\
\end{align*} \]

\[ \begin{align*} 
\dot{x} &= Ax + Bu \\
y &= Cx + Du \\
\end{align*} \]

where:

- \( A \) – system matrix (\( n \times n \))
- \( B \) – input matrix (\( n \times m \))
- \( C \) – output matrix (\( p \times n \))
- \( D \) – feedthrough matrix (\( p \times m \))
From State-Space to Transfer Function

Let us find the *transfer function* from $u$ to $y$ corresponding to the state-space model

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

- in the scalar case ($x, y, u \in \mathbb{R}$), we took the Laplace transform
- the same idea here when working with vectors: just do it component by component
From State-Space to Transfer Function

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

\[ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix} \]

Recall matrix-vector multiplication:

\[ \dot{x}_i = (Ax)_i + (Bu)_i \]
\[ = \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k \]

\[ y_\ell = (Cx)_\ell + (Du)_\ell \]
\[ = \sum_{j=1}^{n} c_{\ell j} x_j + \sum_{k=1}^{m} d_{\ell k} u_k \]
Now we take the Laplace transform:

\[
\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k
\]

\[
\downarrow \mathcal{L}
\]

\[
sX_i(s) - x_i(0) = \sum_{j=1}^{n} a_{ij} X_j(s) + \sum_{k=1}^{m} b_{ik} U_k(s), \quad i = 1, \ldots, n
\]

Write down in matrix-vector form:

\[
sX(s) - x(0) = AX(s) + BU(s)
\]

\[
(Is - A)X(s) = x(0) + BU(s) \quad (I is the n \times n identity matrix)
\]

\[
X(s) = (Is - A)^{-1} x(0) + (Is - A)^{-1} BU(s)
\]
From State-Space to Transfer Function

\[ y_\ell = \sum_{j=1}^{n} c_{\ell j} x_j + \sum_{k=1}^{m} d_{\ell k} u_k \]

\[ \downarrow \mathcal{L} \]

\[ Y_\ell(s) = \sum_{j=1}^{n} c_{\ell j} X_j(s) + \sum_{k=1}^{m} d_{\ell k} U_k(s), \quad \ell = 1, \ldots, p \]

Write down in matrix-vector form:

\[ Y(s) = CX(s) + DU(s) \]
\[ = C \left[(Is - A)^{-1} x(0) + (Is - A)^{-1} BU(s)\right] + DU(s) \]
\[ = C(Is - A)^{-1} x(0) + \left[C(Is - A)^{-1} B + D\right] U(s) \]

To find the input-output t.f., set the IC to 0:

\[ Y(s) = G(s) U(s), \quad \text{where } G(s) = C(Is - A)^{-1} B + D \]
From State-Space to Transfer Function

The transfer function from $u$ to $y$, corresponding to

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is given by

$$G(s) = C(Is - A)^{-1}B + D$$

Observe that $G(s)$ contains information about the state-space matrices $A, B, C, D$!!
From State-Space to Transfer Function

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

\[
Y(s) = G(s)U(s) = [C(Is - A)^{-1}B + D] U(s)
\]

**Important!!**

- \(G(s)\) is *undefined* when the \(n \times n\) matrix \(Is - A\) is *singular* (or noninvertible), i.e., precisely when \(\det(Is - A) = 0\)
- since \(A\) is \(n \times n\), \(\det(Is - A)\) is a *polynomial* of degree \(n\) (the *characteristic polynomial* of \(A\)):

\[
\det(Is - A) = \det\begin{pmatrix}
s - a_{11} & -a_{12} & \cdots & -a_{1n} \\
-a_{21} & s - a_{22} & \cdots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n2} & \cdots & s - a_{nn}
\end{pmatrix},
\]

and its roots are the *eigenvalues* of \(A\)
- \(G\) is (open-loop) stable if all eigenvalues of \(A\) lie in LHP.
Example: Computing $G(s)$

Consider the state-space model in Controller Canonical Form (CCF)*:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} +
\begin{pmatrix}
0 \\
1
\end{pmatrix} u,
\quad y =
\begin{pmatrix}
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

— this is a single-input, single-output (SISO) system, since $u, y \in \mathbb{R}$; the state is two-dimensional.

Let’s compute the transfer function:

\[
G(s) = C(I s - A)^{-1} B \quad (D = 0 \text{ here})
\]

\[
I s - A = \begin{pmatrix}
s & -1 \\
6 & s + 5
\end{pmatrix}
\]

* We will explain this terminology later.
Example: Computing $G(s)$

$$Is - A = \begin{pmatrix} s & -1 \\ 6 & s + 5 \end{pmatrix}$$

— how do we compute $(Is - A)^{-1}$?

A useful formula for the inverse of a $2 \times 2$ matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det M \neq 0 \quad \implies \quad M^{-1} = \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Applying the formula, we get

$$(Is - A)^{-1} = \frac{1}{\det(Is - A)} \begin{pmatrix} s + 5 & 1 \\ -6 & s \end{pmatrix} = \frac{1}{s^2 + 5s + 6} \begin{pmatrix} s + 5 & 1 \\ -6 & s \end{pmatrix}$$
Example: Computing $G(s)$

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} +
\begin{pmatrix}
0 \\
1
\end{pmatrix} u,
\quad
y =
\begin{pmatrix}
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

\[
G(s) = C( Is - A)^{-1} B
\]
\[
= (1 \ 1) \frac{1}{s^2 + 5s + 6} \begin{pmatrix}
s + 5 & 1 \\
-6 & s
\end{pmatrix} \begin{pmatrix}
0 \\
1
\end{pmatrix}
\]
\[
= \frac{1}{s^2 + 5s + 6} \begin{pmatrix}
1 & 1
\end{pmatrix} \begin{pmatrix}
s + 1 \\
s
\end{pmatrix}
\]
\[
= \frac{s + 1}{s^2 + 5s + 6}
\]

\[\blacktriangleleft\] the above state-space model is a \textit{realization} of this t.f.
\[\blacktriangleleft\] note how coefficients 5 and 6 appear in both $G(s)$ and $A$!!
State-Space Realizations of Transfer Functions

\[
\begin{align*}
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} &= 
\begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + 
\begin{pmatrix}
0 \\
1
\end{pmatrix} u,
\end{align*}
\]

\[y = 
\begin{pmatrix}
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

\[G(s) = \frac{s + 1}{s^2 + 5s + 6}
\]

— at least in this example, information about the state-space model \((A, B, C)\) is contained in \(G(s)\).

Is this information recoverable? — i.e., is there only one state-space realization of a given t.f.? Or are there many?

**Answer:** There are infinitely many!
State-Space Realizations of Transfer Functions

Start with

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} +
\begin{pmatrix}
0 \\
1
\end{pmatrix} u,
\quad y =
\begin{pmatrix}
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

and consider a new state-space model

\[
\dot{x} = \bar{A} x + \bar{B} u,
\quad y = \bar{C} x
\]

with

\[
\bar{A} = A^T = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix},
\quad \bar{B} = C^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\quad \bar{C} = B^T = (0 \ 1)
\]

This is a different state-space model!
State-Space Realizations of Transfer Functions

Claim: The state-space model

\[ \dot{x} = \bar{A}x + \bar{B}u, \quad y = \bar{C}x \]

with

\[ \bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{C} = B^T \]

has the same transfer function as the original model with \((A, B, C)\).

Proof:

\[
\bar{C}(Is - \bar{A})^{-1}\bar{B} = B^T (Is - A^T)^{-1} C^T \\
= B^T [(Is - A)^T]^{-1} C^T \\
= B^T [(Is - A)^{-1}]^T C^T \\
= [C(Is - A)^{-1}B]^T \\
= C(Is - A)^{-1}B
\]
State-Space Realizations of Transfer Functions

The state-space model

\[ \dot{x} = \bar{A}x + \bar{B}u, \quad y = \bar{C}x \]

with

\[ \bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{C} = B^T \]

has the same transfer function as the original model with \((A, B, C)\).

But the state-space model is now in the Observer Canonical Form (OCF):

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
0 & -6 \\
1 & -5
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
1 \\
1
\end{pmatrix} u, \quad y = \begin{pmatrix}
0 & 1
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]
Even More Realizations ...

Yet another realization of $G(s) = \frac{s + 1}{s^2 + 5s + 6}$ can be extracted from the partial-fractions decomposition:

$$
G(s) = \frac{s + 1}{(s + 2)(s + 3)} = \frac{2}{s + 3} - \frac{1}{s + 2}.
$$

This is the Modal Canonical Form (MCF):

$$
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix}
= \begin{pmatrix}
-3 & 0 \\
0 & -2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
+ \begin{pmatrix}
1 \\
1
\end{pmatrix} u,

y = \begin{pmatrix}
2 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
$$

Then

$$
C(Is - A)^{-1}B = \begin{pmatrix}
2 & -1
\end{pmatrix}
\begin{pmatrix}
s + 3 & 0 \\
0 & s + 2
\end{pmatrix}^{-1}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
2 & -1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{s+3} & 0 \\
0 & \frac{1}{s+2}
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
2 & -1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{s+3} \\
\frac{1}{s+2}
\end{pmatrix}
= \frac{2}{s + 3} - \frac{1}{s + 2}
$$
a given transfer function $G(s)$ can be realized using infinitely many state-space models

certain properties make some realizations preferable to others

one such property is *controllability*
Controllability Matrix

Consider a single-input system \((u \in \mathbb{R})\):

\[
\dot{x} = Ax + Bu, \quad y = Cx \quad x \in \mathbb{R}^n
\]

The **Controllability Matrix** is defined as

\[
C(A, B) = [B \mid AB \mid A^2B \mid \ldots \mid A^{n-1}B]
\]

— recall that \(A\) is \(n \times n\) and \(B\) is \(n \times 1\), so \(C(A, B)\) is \(n \times n\);
— the controllability matrix only involves \(A\) and \(B\), not \(C\)

We say that the above system is **controllable** if its controllability matrix \(C(A, B)\) is **invertible**.

(This definition is only true for the single-input case; the multiple-input case involves the rank of \(C(A, B)\).)
Controllability Matrix

Consider a single-input system \( (u \in \mathbb{R}) \):

\[
\dot{x} = Ax + Bu, \quad y = Cx
\]

\( x \in \mathbb{R}^n \)

The Controllability Matrix is defined as

\[
C(A, B) = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}
\]

We say that the above system is controllable if its controllability matrix \( C(A, B) \) is invertible.

- As we will see later, if the system is controllable, then we may assign arbitrary closed-loop poles by state feedback of the form \( u = -Kx \).
- Whether or not the system is controllable depends on its state-space realization.
Example: Computing $C(A, B)$

Let’s get back to our old friend:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 \\ 1 \\ \hline C \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2 \times 2} \implies C(A, B) \in \mathbb{R}^{2 \times 2}$

$$C(A, B) = \begin{bmatrix} B & AB \end{bmatrix} \quad AB = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$$\implies C(A, B) = \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix}$$

Is this system controllable?

$$\det C = -1 \neq 0 \implies \text{system is controllable}$$
Controller Canonical Form

A single-input state-space model

\[ \dot{x} = Ax + Bu, \quad y = Cx \]

is said to be in **Controller Canonical Form (CCF)** if the matrices \( A, B \) are of the form

\[
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
* & * & * & \ldots & * & * \\
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1 \\
\end{pmatrix}
\]

A system in CCF is always controllable!!

(The proof of this for \( n > 2 \) uses the Jordan canonical form, we will not worry about this.)
CCF with Arbitrary Zeros

In our example, we had \( G(s) = \frac{s + 1}{s^2 + 5s + 6} \), with a minimum-phase zero at \( z = -1 \).

Let’s consider a general zero location \( s = z \):

\[
G(s) = \frac{s - z}{s^2 + 5s + 6}
\]

This gives us a CCF realization

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
0 \\
1
\end{pmatrix} u,
\quad y = \begin{pmatrix}
-z & 1
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

Since \( A, B \) are the same, \( C(A, B) \) is the same \( \implies \) the system is still controllable.

A system in CCF is controllable for any locations of the zeros.