Plan of the Lecture

- **Review:** Nyquist stability criterion
- **Today’s topic:** Nyquist stability criterion (more examples); phase and gain margins from Nyquist plots.

**Goal:** explore more examples of the Nyquist criterion in action.

**Reading:** FPE, Chapter 6
Consider an arbitrary transfer function $H$.

**Nyquist plot:** $\text{Im } H(j\omega)$ vs. $\text{Re } H(j\omega)$ as $\omega$ varies from $-\infty$ to $\infty$. 

![Nyquist Plot Diagram]
**Goal:** count the number of RHP poles (if any) of the closed-loop transfer function

\[
\frac{KG(s)}{1 + KG(s)}
\]

based on frequency-domain characteristics of the plant transfer function \( G(s) \)
The Nyquist Theorem

Nyquist Theorem (1928) Assume that $G(s)$ has no poles on the imaginary axis*, and that its Nyquist plot does not pass through the point $-1/K$. Then

$$N = Z - P$$

$$\#(\text{ccw of } -1/K \text{ by Nyquist plot of } G(s))$$

$$= \#(\text{RHP closed-loop poles}) - \#(\text{RHP open-loop poles})$$

* Easy to fix: draw an infinitesimally small circular path that goes \emph{around} the pole and stays in RHP.
Nyquist Stability Criterion. Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain $K$) is stable if and only if the Nyquist plot of $G(s)$ encircles the point $-1/K$ $P$ times counterclockwise, where $P$ is the number of unstable (RHP) open-loop poles of $G(s)$.
Applying the Nyquist Criterion

Workflow:

\[ \text{Bode } M \text{ and } \phi\text{-plots} \quad \rightarrow \quad \text{Nyquist plot} \]

Advantages of Nyquist over Routh–Hurwitz

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- less computational, more geometric (came 55 years after Routh)
Example 1 (From Last Lecture)

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]  
(no open-loop RHP poles)

Characteristic equation:

\[ (s + 1)(s + 2) + K = 0 \quad \iff \quad s^2 + 3s + K + 2 = 0 \]

From Routh, we already know that the closed-loop system is stable for \( K > -2 \).

We will now reproduce this answer using the Nyquist criterion.
Example 1

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \quad \text{(no open-loop RHP poles)} \]

Strategy:

- Start with the Bode plot of \( G \)
- Use the Bode plot to graph \( \text{Im } G(j\omega) \) vs. \( \text{Re } G(j\omega) \) for \( 0 \leq \omega < \infty \)
- This gives only a portion of the entire Nyquist plot
  \[(\text{Re } G(j\omega), \text{Im } G(j\omega)) , -\infty < \omega < \infty \]

- Symmetry:
  \[ G(-j\omega) = \overline{G(j\omega)} \]
  — Nyquist plots are always symmetric w.r.t. the real axis!!
Example 1

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]

(no open-loop RHP poles)

Bode plot:

Nyquist plot:
Example 1: Applying the Nyquist Criterion

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \] (no open-loop RHP poles)

Nyquist plot:

\[ \#(\text{☉ of } -1/K) = \#(\text{RHP CL poles}) - \#(\text{RHP OL poles}) = 0 \]

\[ \implies K \in \mathbb{R} \text{ is stabilizing if and only if } \#(\text{☉ of } -1/K) = 0 \]

- If \( K > 0 \), \( \#(\text{☉ of } -1/K) = 0 \)
- If \( 0 < -1/K < 1/2 \),
  \[ \#(\text{☉ of } -1/K) > 0 \implies \text{closed-loop stable for } K > -2 \]
Example 2

\[ G(s) = \frac{1}{(s - 1)(s^2 + 2s + 3)} = \frac{1}{s^3 + s^2 + s - 3} \]

\#(RHP open-loop poles) = 1 at \( s = 1 \)

**Routh:** the characteristic polynomial is

\[ s^3 + s^2 + s + K - 3 \ — 3rd \ degree \]

— stable if and only if \( K - 3 > 0 \) and \( 1 > K - 3 \).

Stability range: \( 3 < K < 4 \)

Let’s see how to spot this using the Nyquist criterion ...
Example 2

\[ G(s) = \frac{1}{(s - 1)(s^2 + 2s + 3)} \]

(1 open-loop RHP pole)

Bode plot:

Nyquist plot:

\[ \omega = 0 \quad M = 1/3, \quad \phi = -180^\circ \]
\[ \omega = 1 \quad M = 1/4, \quad \phi = -180^\circ \]
\[ \omega \to \infty \quad M \to 0, \quad \phi \to -270^\circ \]
Example 2: Applying the Nyquist Criterion

\[ G(s) = \frac{1}{(s - 1)(s^2 + 2s + 3)} \]  

(1 open-loop RHP pole)

Nyquist plot:

\[
#(\bigcirc \text{ of } -1/K) = #(RHP \text{ CL poles}) - #(RHP \text{ OL poles}) \]

\[ = 1 \]

\[ K \in \mathbb{R} \text{ is stabilizing if and only if} \]

\[ #(\bigcirc \text{ of } -1/K) = -1 \]

Which points \(-1/K\) are encircled once \(\bigcirc\) by this Nyquist plot?

Only \(-1/3 < -1/K < -1/4\) \[ \implies 3 < K < 4 \]
Example 2: Nyquist Criterion and Phase Margin

Closed-loop stability range for \( G(s) = \frac{1}{(s - 1)(s^2 + 2s + 3)} \) is \( 3 < K < 4 \) (using either Routh or Nyquist).

We can interpret this in terms of phase margin:

So, in this case, stability \( \iff \) PM > 0 (typical case).
Example 3

\[
G(s) = \frac{s - 1}{(s + 2)(s^2 - s + 1)} = \frac{s - 1}{s^3 + s^2 - s + 2}
\]

Open-loop poles:

\[
s = -2 \quad \text{(LHP)}
\]

\[
s^2 - s + 1 = 0
\]

\[
(s - \frac{1}{2})^2 + \frac{3}{4} = 0
\]

\[
s = \frac{1}{2} \pm j \frac{\sqrt{3}}{2} \quad \text{(RHP)}
\]

∴ 2 RHP poles
Example 3

\[ G(s) = \frac{s - 1}{(s + 2)(s^2 - s + 1)} = \frac{s - 1}{s^3 + s^2 - s + 2} \]

Routh:

char. poly. \[ s^3 + s^2 - s + 2 + K(s - 1) \]
\[ s^2 + s^2 + (K - 1)s + 2 - K \quad (3\text{rd-order}) \]

— stable if and only if

\[ K - 1 > 0 \]
\[ 2 - K > 0 \]
\[ K - 1 > 2 - K \]

— stability range is \( \frac{3}{2} < K < 2 \)
Example 3

\[ G(s) = \frac{s - 1}{(s + 2)(s^2 - s + 1)} \]

(2 open-loop RHP poles)

\[ \phi = 180^\circ \text{ when:} \]

\[ \begin{align*}
\triangleright & \quad \omega = 0 \text{ and } \omega \to 0 \\
\triangleright & \quad \omega = 1 / \sqrt{2}:
\end{align*} \]

\[ \left. \frac{j\omega - 1}{(j\omega - 1)((j\omega)^2 - j\omega + 1)} \right|_{\omega = 1 / \sqrt{2}} = \frac{j / \sqrt{2} - 1}{\left( \frac{j}{\sqrt{2}} + 2 \right) \left( -\frac{1}{2} - \frac{j}{\sqrt{2}} + 1 \right)} = \frac{2}{3} \] 

(need to guess this, e.g., by mouseclicking in Matlab)
Example 3

\[ G(s) = \frac{s - 1}{s^3 + s^2 - s + 2} \]

(2 open-loop RHP poles)

Bode plot:

Nyquist plot:

\[
\begin{align*}
\omega &= 0 & M &= 1/2, \phi = 180^\circ \\
\omega &= 1/\sqrt{2} & M &= 2/3, \phi = 180^\circ \\
\omega &\to \infty & M &\to 0, \phi &\to 180^\circ
\end{align*}
\]
Example 3: Applying the Nyquist Criterion

\[ G(s) = \frac{s - 1}{s^3 + s^2 - s + 2} \]

(2 open-loop RHP poles)

Nyquist plot:

\[ K \in \mathbb{R} \text{ is stabilizing if and only if} \]

\[ \#(\circlearrowleft \text{ of } -1/K) = -2 \]

Which points \(-1/K\) are encircled twice \(\circlearrowleft\) by this Nyquist plot?

\[
\#(\circlearrowleft \text{ of } -1/K)
= \#(\text{RHP CL poles})
- \#(\text{RHP OL poles})
= 2
\]

only \(-2/3 < -1/K < -1/2\)

\[ \Rightarrow \frac{3}{2} < K < 2 \]
Example 2: Nyquist Criterion and Phase Margin

CL stability range for \( G(s) = \frac{s - 1}{s^3 + s^2 - s + 2} : K \in (3/2, 2) \)

We can interpret this in terms of phase margin:

So, in this case, stability \( \iff \) PM < 0 (atypical case; Nyquist criterion is the only way to resolve this ambiguity of Bode plots).
Stability Margins

How do we determine stability margins (GM & PM) from the Nyquist plot?

GM & PM are defined relative to a given $K$, so consider Nyquist plot of $KG(s)$ (we only draw the $\omega > 0$ portion):

How do we spot GM & PM?

- GM = $1/M_{180^\circ}$
  
  — if we divide $K$ by $M_{180^\circ}$, then the Nyquist plot will pass through $(-1, 0)$, giving $M = 1, \phi = 180^\circ$

- PM = $\phi$
  
  — the phase difference from $180^\circ$ when $M = 1$