Plan of the Lecture

- **Review**: control design using frequency response
- **Today’s topic**: Nyquist stability criterion

**Goal**: learn how to detect the presence of RHP poles of the closed-loop transfer function as the gain $K$ is varied using frequency-response data

**Reading**: FPE, Chapter 6
Review: Frequency Domain Design Method

Design based on Bode plots is good for:

- easily visualizing the concepts
- evaluating the design and seeing which way to change it
- using experimental data (frequency response of the uncontrolled system can be measured experimentally)
Design based on Bode plots is not good for:

- exact closed-loop pole placement (root locus is more suitable for that)
- deciding if a given $K$ is stabilizing or not ...
  - we can only measure \textit{how far} we are from instability (using GM or PM), if we know that we are stable
  - however, we don’t have a way of checking whether a given $K$ is stabilizing from frequency response data

What we want is a frequency-domain substitute for the Routh–Hurwitz criterion — this is the \textbf{Nyquist criterion}, which we will discuss in today’s lecture.
Nyquist Stability Criterion

Goal: count the number of RHP poles (if any) of the closed-loop transfer function

\[
\frac{KG(s)}{1 + KG(s)}
\]

based on frequency-domain characteristics of the plant transfer function \(G(s)\)
Review: Nyquist Plot

Consider an arbitrary *strictly proper* transfer function $H$:

$$H(s) = \frac{(s - z_1) \ldots (s - z_m)}{(s - p_1) \ldots (s - p_n)}, \quad m < n$$

Nyquist plot: $\text{Im } H(j\omega)$ vs. $\text{Re } H(j\omega)$ as $\omega$ varies from $-\infty$ to $\infty$
Nyquist Plot as a Mapping of the $s$-Plane

We can view the Nyquist plot of $H$ as the image of the imaginary axis $\{j\omega : -\infty < \omega < \infty\}$ under the mapping $H : \mathbb{C} \rightarrow \mathbb{C}$.
Transformation of a Closed Contour Under $H$

If we choose any closed curve (or contour) $C$ on the left, it will get mapped by $H$ to some other curve (contour) on the right:

Important: when working with contours in the complex plane, always keep track of the direction in which we traverse the contour (clockwise vs. counterclockwise)!!
Phase of $H$ Along a Contour

For any $s \in \mathbb{C}$, the phase (or argument) of $H(s)$ is

$$\angle H(s) = \angle \frac{(s - z_1) \ldots (s - z_m)}{(s - p_1) \ldots (s - p_n)}$$

$$= \sum_{i=1}^{m} \angle(s - z_i) - \sum_{j=1}^{n} \angle(s - p_j)$$

$$= \sum_{i=1}^{m} \psi_i - \sum_{j=1}^{n} \varphi_j$$

We are interested in how $\angle H(s)$ changes as $s$ traverses a closed, clockwise ($\mathcal{C}$) oriented contour $C$ in the complex plane.

We will look at several cases, depending on how the contour is located relative to poles and zeros of $H$. 
Case 1: Contour Encircles a Zero

Suppose that $C$ is a closed, $⟳$-oriented contour in $\mathbb{C}$ that encircles a zero of $H(s)$:

How does $\angle H(s)$ change as we go around $C$?
Case 1: Contour Encircles a Zero

How does $\angle H(s)$ change as we go around $C$?

Let’s see what happens to angles from $s$ to poles/zeros of $H$:

- $\varphi_1$ and $\varphi_2$ return to their original values
- $\psi_1$ picks up a net change of $-360^\circ$
- therefore, $\angle H(s)$ picks up a net change of $-360^\circ$, so $H(C)$ encircles the origin once, clockwise ($\circlearrowright$)
Case 2: Contour Encircles a Pole

Suppose that $C$ is a closed, $\bigcirc$-oriented contour in $\mathbb{C}$ that encircles a pole of $H(s)$:

How does $\angle H(s)$ change as we go around $C$?
Case 2: Contour Encircles a Pole

How does $\angle H(s)$ change as we go around $C$?

Let’s see what happens to angles from $s$ to poles/zeros of $H$:

- $\varphi_1$ and $\psi_1$ return to their original values
- $\varphi_2$ picks up a net change of $-360^\circ$
- therefore, $\angle H(s)$ picks up a net change of $360^\circ$, so $H(C)$ encircles the origin once counterclockwise ($\bigcirc$)
Case 3: Contour Encircles No Poles or Zeros

Suppose that \( C \) is a closed, \( \mathbb{C} \)-oriented contour in \( \mathbb{C} \) that does not encircle any poles or zeros of \( H(s) \):

\[
\begin{array}{c}
\text{Im } s \\
\text{Re } s
\end{array}
\]

How does \( \angle H(s) \) change as we go around \( C \)?
How does $\angle H(s)$ change as we go around $C$?

Let’s see what happens to angles from $s$ to poles/zeros of $H$:

- $\varphi_1, \varphi_2, \psi_1$ all return to their original values
- therefore, no net change in $\angle H(s)$, so $H(C)$ does not encircle the origin
The Argument Principle

These special cases all lead to the following general result:

The Argument Principle. Let $C$ be a closed, clockwise $\circlearrowleft$ oriented contour not passing through any zeros or poles* of $H(s)$. Let $H(C)$ be the image of $C$ under the map $s \mapsto H(s)$:

$$H(C) = \{H(s) : s \in \mathbb{C}\}.$$ 

Then:

$$\#(\text{clockwise encirclements } \circlearrowleft \text{ of } 0 \text{ by } H(C))$$

$$= \#(\text{zeros of } H(s) \text{ inside } C) - \#(\text{poles of } H(S) \text{ inside } C).$$

More succinctly,

$$N = Z - P$$

* will see the reason for this later ...
The Argument Principle

\[ N = Z - P \]

- If \( N < 0 \), it means that \( H(C) \) encircles the origin counterclockwise (\( \odot \)).
- We do not want \( C \) to pass through any pole of \( H \) because then \( H(C) \) would not be defined.
- We also do not want \( C \) to pass through any zero of \( H \) because then \( 0 \in H(C) \), so \( \#(\text{encirclements}) \) is not well-defined.
We are interested in RHP poles, so let’s choose a suitable contour $C$ that encloses the RHP:

From now on, $C =$ imaginary axis plus the “path around infinity.”

If $H$ is strictly proper, then $H(\infty) = 0.$
From Argument Principle to Nyquist Criterion

Harry Nyquist
(1889–1976)

With this choice of \( C \),

\[ H(C) = \text{Nyquist plot of } H \]

(image of the imaginary axis under the map
\( H : \mathbb{C} \rightarrow \mathbb{C} \); if \( H \) is strictly proper, \( 0 = H(\infty) \))
We are interested in RHP roots of $1 + KG(s)$, where $G$ is the plant transfer function.

Thus, we choose $H(s) = 1 + KG(s)$
We now examine the Nyquist plot of \( H(s) = 1 + KG(s) \).

By the argument principle,

\[
N = Z - P,
\]

where \( N \) = \#(∩ encirclements of 0 by Nyquist plot of \( 1 + KG(s) \)),

\( Z \) = \#(zeros of \( 1 + KG(s) \) inside \( C \)),

\( P \) = \#(poles of \( 1 + KG(s) \) inside \( C \)).

Now we extract information about RHP roots of \( 1 + KG(s) \).
Nyquist Criterion: $N$

$$N = \#(\bigcirc \text{ encirclements of } 0 \text{ by Nyquist plot of } 1 + KG(s))$$
$$= \#(\bigcirc \text{ encirclements of } -1 \text{ by Nyquist plot of } KG(s))$$
$$= \#(\bigcirc \text{ encirclements of } -1/K \text{ by Nyquist plot of } G(s))$$

— can be read off the Nyquist plot of the open-loop t.f. $G!!$
Nyquist Criterion: \( Z \)

\[ G(s) = \frac{q(s)}{p(s)}, \quad \text{deg}(q) \leq \text{deg}(p) \]

\[ 1 + KG(s) = \frac{p(s) + Kq(s)}{p(s)} \]

closed-loop t.f. \( = \frac{KG(s)}{1 + KG(s)} = \frac{Kq(s)}{p(s) + Kq(s)} \)

Therefore:

\[ Z = \#(\text{zeros of } 1 + KG(s) \text{ inside } C) \]

\[ = \#(\text{RHP zeros of } 1 + KG(s)) \]

\[ = \#(\text{RHP closed-loop poles}) \]
Nyquist Criterion: $P$

\[
G(s) = \frac{q(s)}{p(s)}, \quad \text{deg}(q) \leq \text{deg}(p)
\]

\[
1 + KG(s) = 1 + K \frac{q(s)}{p(s)} = \frac{p(s) + Kq(s)}{p(s)}
\]

Therefore:

\[
P = \#(\text{poles of } 1 + KG(s) \text{ inside } C)
\]
\[
= \#(\text{RHP poles of } 1 + KG(s))
\]
\[
= \#(\text{RHP roots of } p(s))
\]
\[
= \#(\text{RHP open-loop poles})
\]
Nyquist Theorem (1928) Assume that $G(s)$ has no poles on the imaginary axis*, and that its Nyquist plot does not pass through the point $-1/K$. Then

$$N = Z - P$$

$$\#(\text{circle of } -1/K \text{ by Nyquist plot of } G(s))$$

$$= \#(\text{RHP closed-loop poles}) - \#(\text{RHP open-loop poles})$$

* Easy to fix: draw an infinitesimally small circular path that goes around the pole and stays in RHP
The Nyquist Stability Criterion

Nyquist Stability Criterion. Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain $K$) is stable if and only if the Nyquist plot of $G(s)$ encircles the point $-1/K\,P$ times counterclockwise, where $P$ is the number of unstable (RHP) open-loop poles of $G(s)$. 

\[
\begin{align*}
N &= \text{(○ of } -1/K) \\
Z &= \text{(unstable CL poles)} - \text{(unstable OL poles)} \\
Z &= N + P \\
Z &= 0 \implies N = -P
\end{align*}
\]
Applying the Nyquist Criterion

Workflow:

Bode $M$ and $\phi$-plots $\rightarrow$ Nyquist plot

Advantages of Nyquist over Routh–Hurwitz

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- less computational, more geometric (came 55 years after Routh)
Example

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \quad \text{(no open-loop RHP poles)} \]

Characteristic equation:

\[ (s + 1)(s + 2) + K = 0 \quad \iff \quad s^2 + 3s + K + 2 = 0 \]

From Routh, we already know that the closed-loop system is stable for \( K > -2 \).

We will now reproduce this answer using the Nyquist criterion.
Example

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]  
(no open-loop RHP poles)

Strategy:

- Start with the Bode plot of \( G \)
- Use the Bode plot to graph \( \text{Im } G(j\omega) \) vs. \( \text{Re } G(j\omega) \) for \( 0 \leq \omega < \infty \)
- This gives only a portion of the entire Nyquist plot:
  \[ (\text{Re } G(j\omega), \text{Im } G(j\omega)), \quad -\infty < \omega < \infty \]

- Symmetry:
  \[ G(-j\omega) = \overline{G(j\omega)} \]

---

Nyquist plots are always symmetric w.r.t. the real axis!!
Example

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]

(no open-loop RHP poles)

Bode plot:

Nyquist plot:
Example: Applying the Nyquist Criterion

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \quad \text{(no open-loop RHP poles)} \]

Nyquist plot:

\#(\circ \text{ of } -1/K) = \#(\text{RHP CL poles}) - \#(\text{RHP OL poles}) = 0

\implies K \in \mathbb{R} \text{ is stabilizing if and only if}

\#(\circ \text{ of } -1/K) = 0

- If \( K > 0 \), \#(\circ \text{ of } -1/K) = 0
- If \( 0 < -1/K < 1/2 \), \#(\circ \text{ of } -1/K) > 0 \implies \text{closed-loop stable for } K > -2