Plan of the Lecture

▶ Review: Bode plots for three types of transfer functions
▶ Today’s topic: stability from frequency response; gain and phase margins

Goal: learn to read off stability properties of the closed-loop system from the Bode plot of the open-loop transfer function; define and calculate Gain and Phase Margins, important quantitative measures of “distance to instability.”

Reading: FPE, Section 6.1
Stability from Frequency Response

Consider this unity feedback configuration:

\[ G(s)Y + RK \]

**Question:** How can we decide whether the *closed-loop* system is stable for a given value of \( K > 0 \) based on our knowledge of the *open-loop* transfer function \( KG(s) \)?
Stability from Frequency Response

Question: How can we decide whether the closed-loop system is stable for a given value of $K > 0$ based on our knowledge of the open-loop transfer function $KG(s)$?

One answer: use root locus.

Points on the root locus satisfy the characteristic equation

$$1 + KG(s) = 0 \iff KG(s) = -1 \iff G(s) = -\frac{1}{K}$$

If $s \in \mathbb{C}$ is on the RL, then

$$|KG(s)| = 1 \quad \text{and} \quad \angle KG(s) = \angle G(s) = 180^\circ \mod 360^\circ$$
Stability from Frequency Response

\[ G(s) Y + R K G(s) \]

Question: How can we decide whether the closed-loop system is stable for a given value of \( K > 0 \) based on our knowledge of the open-loop transfer function \( KG(s) \)?

Another answer: let’s look at the Bode plots:

\[ \omega \mapsto |KG(j\omega)| \quad \text{on log-log scale} \]
\[ \omega \mapsto \angle KG(j\omega) \quad \text{on log-linear scale} \]

— Bode plots show us magnitude and phase, but only for \( s = j\omega, 0 < \omega < \infty \)

How does this relate to the root locus? \( j\omega\)-crossings!!
Stability from Frequency Response

Stability from frequency response. If \( s = j\omega \) is on the root locus (for some value of \( K \)), then

\[
|KG(j\omega)| = 1 \quad \text{and} \quad \angle KG(j\omega) = 180^\circ \mod 360^\circ
\]

Therefore, the transition from stability to instability can be detected in two different ways:

- from root locus — as \( j\omega \)-crossings
- from Bode plots — as \( M = 1 \) and \( \phi = 180^\circ \) at some frequency \( \omega \) (for a given value of \( K \))
Example

\[ KG(s) = \frac{K}{s(s^2 + 2s + 2)} \]

Characteristic equation:

\[ 1 + \frac{K}{s(s^2 + 2s + 2)} = 0 \]
\[ s(s^2 + 2s + 2) + K = 0 \]
\[ s^3 + 2s^2 + 2s + K = 0 \]

Recall the necessary & sufficient condition for stability for a 3rd-degree polynomial \( s^3 + a_1 s^2 + a_2 s + a_3 \):

\[ a_1, a_2, a_3 > 0, \quad a_1 a_2 > a_3. \]

Here, the closed-loop system is stable if and only if \( 0 < K < 4 \).

Let’s see what we can read off from the Bode plots.
Example, continued

\[ KG(s) = \frac{K}{s(s^2 + 2s + 2)} \]

Bode form: \[ KG(j\omega) = \frac{K}{2j\omega \left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)} \]

Plot the magnitude first:

- Type 1 (low-frequency) asymptote: \[ \frac{K/2}{j\omega} \]
  \[ K_0 = K/2, \ n = -1 \implies \text{slope} = -1, \text{ passes through } (\omega = 1, M = K/2) \]

- Type 3 (complex pole) asymptote:
  break-point at \( \omega = \sqrt{2} \implies \text{slope down by 2} \]

- \( \zeta = \frac{1}{\sqrt{2}} \implies \text{no resonant peak} \)
Example, Magnitude Plot

\[ KG(j\omega) = \frac{K}{2j\omega \left( (\frac{j\omega}{\sqrt{2}})^2 + j\omega + 1 \right)} \]

Magnitude plot for \( K = 4 \) (the critical value):

When \( \omega = \sqrt{2} \), \( M = |4G(j\omega)| = \left| \frac{2}{j\sqrt{2} \left( j^2 + j\sqrt{2} + 1 \right)} \right| = 1 \)
Example, Phase Plot

\[ KG(j\omega) = \frac{K}{2j\omega \left( \left( \frac{j\omega}{\sqrt{2}} \right)^2 + j\omega + 1 \right)} \]

Phase plot (independent of \( K \)): 

When \( \omega = \sqrt{2}, \phi = -180^\circ \)
For the critical value \( K = 4 \):

\[ M = 1 \text{ and } \phi = 180^\circ \mod 360^\circ \text{ at } \omega = \sqrt{2} \]
Crossover Frequency and Stability

**Definition:** The frequency at which $M = 1$ is called the **crossover frequency** and denoted by $\omega_c$.

Transition from stability to instability on the Bode plot:

for critical $K$, $\angle G(j\omega_c) = 180^\circ$
Effect of Varying $K$

What happens as we vary $K$?

- $\phi$ independent of $K$ $\implies$ only the $M$-plot changes
- If we multiply $K$ by 2:
  \[
  \log(2M) = \log 2 + \log M
  \]
  $M$-plot shifts up by $\log 2$
- If we divide $K$ by 2:
  \[
  \log\left(\frac{1}{2}M\right) = \log \frac{1}{2} + \log M
  \]
  $M$-plot shifts down by $\log 2$

Changing the value of $K$ moves the crossover frequency $\omega_c$!!
Effect of Varying $K$

Changing the value of $K$ moves the crossover frequency $\omega_c$!!

What happens as we vary $K$?

$$\angle KG(j\omega_c) \begin{cases} > -180^\circ, & \text{for } K < 4 \\ = -180^\circ, & \text{for } K = 4 \\ < -180^\circ, & \text{for } K > 4 \end{cases}$$

(stable) 
(critical) 
(unstable)
Effect of Varying $K$

Changing the value of $K$ moves the crossover frequency $\omega_c$!!

Equivalently, we may define $\omega_{180^\circ}$ as the frequency at which

$$\phi = 180^\circ \mod 360^\circ.$$  

Then, in this example*,

$$|KG(j\omega_{180^\circ})| < 1 \iff \text{stability}$$

$$|KG(j\omega_{180^\circ})| > 1 \iff \text{instability}$$

* Not a general rule; conditions will vary depending on the system, must use either root locus or Nyquist plot to resolve ambiguity.
Stability from Frequency Response

Consider this unity feedback configuration:

\[ R \stackrel{+}{\longrightarrow} K \rightarrow G(s) \rightarrow Y \]

Suppose that the closed-loop system, with transfer function

\[ \frac{KG(s)}{1 + KG(s)} \]

is stable for a given value of \( K \).

**Question:** Can we use the Bode plot to determine how far from instability we are?

Two important characteristics: gain margin (GM) and phase margin (PM).
Gain Margin

Back to our example: \[ G(s) = \frac{1}{s(s^2 + 2s + 2)}, \quad K = 2 \text{ (stable)} \]

**Gain margin** (GM) is the factor by which \( K \) can be multiplied before we get \( M = 1 \) when \( \phi = 180^\circ \)

Since varying \( K \) doesn’t change \( \omega_{180^\circ} \), to find GM we need to inspect \( M \) at \( \omega = \omega_{180^\circ} \)
Gain Margin

Our example: \[ G(s) = \frac{1}{s(s^2 + 2s + 2)}, \; K = 2 \text{ (stable)} \]

Gain margin (GM) is the factor by which \( K \) can be multiplied before we get \( M = 1 \) when \( \phi = 180^\circ \).

Since varying \( K \) doesn’t change \( \omega_{180^\circ} = \sqrt{2} \), to find GM we need to inspect \( M \) at \( \omega = \omega_{180^\circ} \).

In this example:

\[ M = 0.5 \cdot (-6 \text{ dB}) \]

at \( \omega_{180^\circ} = \sqrt{2} \)

so \( \text{GM} = 2 \)
Phase Margin

Our example: \( G(s) = \frac{1}{s(s^2 + 2s + 2)}, \ K = 2 \) (stable)

Phase margin (PM) is the amount by which the phase at the crossover frequency \( \omega_c \) differs from 180° mod 360°

To find PM, we need to inspect \( \phi \) at \( \omega = \omega_c \)

In this example:

\[
\phi = -148^\circ
\]

so \( PM = (-148^\circ) - (-180^\circ) = 32^\circ \)

(in practice, want PM \( \geq 30^\circ \))
Example 2

\[
G(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s} \quad \zeta, \omega_n > 0
\]

Consider gain \( K = 1 \), which gives closed-loop transfer function

\[
\frac{KG(s)}{1 + KG(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s} \quad \text{prototype 2nd-order response}
\]

Question: what is the gain margin at \( K = 1 \)?

Answer: \( \text{GM} = \infty \)
Example 2

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)} \]

Let’s look at the phase plot:

- starts at \(-90^\circ\) (Type 1 term with \(n = -1\))
- goes down by \(-90^\circ\) (Type 2 pole)

Recall: to find GM, we first need to find \(\omega_{180^\circ}\), and here there is no such \(\omega \implies \text{no GM} \).
Example 2

So, at \( K = 1 \), the gain margin of

\[
G(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s} = \frac{\omega_n^2}{s(s + 2\zeta \omega_n)}
\]

is equal to \( \infty \) — what does that mean?

It means that we can keep on increasing \( K \) indefinitely without ever encountering instability.

But we already knew that: the characteristic polynomial is

\[
p(s) = s^2 + 2\zeta \omega_n s + \omega_n^2,
\]

which is always stable.

What about phase margin?
Example 2: Phase Margin

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left(\frac{j\omega}{2\zeta\omega_n} + 1\right)} \]

Let’s look at the magnitude plot:

- low-frequency asymptote slope \(-1\) (Type 1 term, \(n = -1\))
- slope down by 1 past the breakpt. \(\omega = 2\zeta\omega_n\) (Type 2 pole)

\[ \Longrightarrow \text{there is a finite crossover frequency} \ \omega_c \]

![Graph showing the magnitude plot with slopes and crossover frequency marked.](image-url)
Example 2: Magnitude Plot

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left(\frac{j\omega}{2\zeta\omega_n} + 1\right)} \]

It can be shown that, for this system,

\[ \left. \text{PM} \right|_{K=1} = \tan^{-1} \left( \frac{2\zeta}{\sqrt{4\zeta^4 + 1 - 2\zeta^2}} \right) \]

— for PM < 70°, a good approximation is PM ≈ 100 \cdot \zeta
Phase Margin for 2nd-Order System

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega \left(\frac{j\omega}{2\zeta\omega_n} + 1\right)} \]

\[ \text{PM}|_{K=1} = \tan^{-1} \left( \frac{2\zeta}{\sqrt{4\zeta^4 + 1 - 2\zeta^2}} \right) \approx 100 \cdot \zeta \]

Conclusions:

larger PM \iff better damping
(open-loop quantity) \hspace{1cm} (closed-loop characteristic)

Thus, the overshoot \( M_p = \exp \left( -\frac{\pi \zeta}{\sqrt{1-\zeta^2}} \right) \) and resonant peak \( M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1 \) are both related to PM through \( \zeta \)!!
In the next lecture, we will see the following more generally:

**Bode’s Gain-Phase Relationship**: all important characteristics of the closed-loop time response can be related to the phase margin of the open-loop transfer function!!

In fact, we will use a quantitative statement of this relationship as a design guideline.