Plan of the Lecture

- **Review**: basic properties and benefits of feedback control
- **Today’s topic**: introduction to Proportional-Integral-Derivative (PID) control

**Goal**: study basic features and capabilities of PID control (industry standard since 1950’s): arbitrary pole placement; reference tracking; disturbance rejection

**Reading**: FPE, Sections 4.1–4.3; lab manual
Recap: Benefits of Feedback Control

From last lecture: feedback control

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

So far, we have only looked at proportional feedback (scalar gain) and 1st-order plants. Now we will add two more basic ingredients and examine their effect on higher-order systems.

We will consider the following plant transfer function:

\[ G(s) = \frac{1}{s^2 - 1} \]

- unstable: poles at \( s = \pm 1 \) (one pole in RHP)
- 2nd-order

- not as easy as DC motor, which was 1st-order and stable.
Proportional Feedback

\[ R \xrightarrow{+} E \xrightarrow{} K_P \xrightarrow{} U \xrightarrow{\frac{1}{s^2 - 1}} Y \]

\[ K_P - \text{“proportional gain” (P-gain)} \quad U = K_P E \]

Let’s try to find a value of \( K_P \) that would stabilize the system:

\[
\frac{Y}{R} = \frac{\frac{K_P}{s^2 - 1}}{1 + \frac{K_P}{s^2 - 1}} = \frac{K_P}{s^2 - 1 + K_P}
\]

— the polynomial in the denominator has zero coefficient of \( s \)

\[ \implies \text{necessary condition for stability is not satisfied.} \]

The feedback system is not stable for any value of \( K_P \)!!
Derivative Feedback

Let’s feed the *derivative of the error*, multiplied by some gain, back into the plant:

Motivation: derivative = rate of change; faster change \(\implies\) more control needed.

Caveat: multiplication by \(s\) is not a causal element (why?)

Derivative action and lack of causality: recall

\[
\dot{e}(t) \approx \frac{e(t + \delta) - e(t)}{\delta} \quad \text{(for small } \delta) \]

— if \(\delta > 0\), \(e(t + \delta)\) is in the future of \(e(t)!!\)
Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx \\
\end{align*}
\]

\[
\begin{align*}
sX &= AX + BU \\
y &= CX \\
\end{align*}
\]

\[
\begin{align*}
(s - A)X &= BU \\
\frac{Y}{U} &= \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)}
\end{align*}
\]

\[\text{deg}(q) < \text{deg}(p) \quad \text{— strictly proper transfer function}\]

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du \\
\end{align*}
\]

\[
\begin{align*}
sX &= AX + BU \\
Y &= CX + DU \\
\end{align*}
\]

\[
\begin{align*}
Y &= \frac{CB}{s - A}U + DU \\
&= \frac{CB + D(s - A)}{s - A}U \equiv \frac{q(s)}{p(s)}
\end{align*}
\]

\[\text{deg}(q) = \text{deg}(p) \quad \text{— proper transfer function}\]

Causal systems have proper transfer functions.
Lack of Causality

But if $u = K \dot{e}$, then $U = K s E \quad \Rightarrow \quad \frac{U}{E} = K s = \frac{q(s)}{p(s)}$

$\deg(q) > \deg(p)$— *improper system* (lack of causality)

So, $E \mapsto K_D s E$ is not implementable directly, but we can implement an approximation, e.g.

$$\frac{K_D a s}{a + s} \rightarrow K_D s \quad \text{as } a \to \infty$$

(this can be done using op-amps).

Alternatively, we can approximate derivative action using finite differences:

$$\dot{e}(t) \approx \frac{e(t + \delta) - e(t)}{\delta},$$

but then we must tolerate delays — must wait until time $t + \delta$ to issue a control signal meant for time $t$. 
Disclaimer 2 about D-Feedback: Noise Amplification

Differentiators amplify noise (noise \(\rightarrow\) rapid changes in the reference).

In the lab, D-feedback is implemented differently, in the feedback path. This way, we avoid differentiating the reference, which may be rapidly changing:

\[
\begin{align*}
Y &= R + K_D s G(s) \\
\frac{Y}{R} &= \frac{K_D s G(s)}{1 + K_D s G(s)} \\
\end{align*}
\]

Before:

\[
\begin{align*}
Y &= G(s) \\
\frac{Y}{R} &= \frac{G(s)}{1 + K_D s G(s)} \\
Poles: &\quad 1 + K_D s G(s) = 0
\end{align*}
\]

— same poles, but different zeros.

Now the reference signal is *smoothed out* by the plant \(G(s)\) before entering the differentiator, which minimizes distortion due to noise.
Back to Analysis: Derivative Feedback

\[
\frac{Y}{R} = \frac{KDs}{s^2 - 1} = \frac{KDs}{s^2 + KDs - 1}
\]

— still not good: the denominator has a negative coefficient
\[\Rightarrow\] not stable; also we have picked up a zero at the origin.

But:

- P-control affected the coefficient of \(s^0\) (constant term)
- D-control affected the coefficient of \(s\)
— let’s combine them!!
Proportional-Derivative (PD) Control

\[ R \rightarrow E \rightarrow K_P + K_D s \rightarrow U \rightarrow \frac{1}{s^2 - 1} \rightarrow Y \]

\[
Y = \frac{K_P + K_D s}{s^2 - 1} = \frac{K_P + K_D s}{1 + \frac{s^2}{s^2 + K_D s + K_P - 1}}
\]

— now, if we set \( K_D > 0 \) and \( K_P > 1 \), then the transfer function will be stable.

**Even more:** by choosing \( K_P \) and \( K_D \), we can *arbitrarily* assign coefficients of the denominator, and therefore the poles of the transfer function:

PD control gives us **arbitrary pole placement**!!
Proportional-Derivative (PD) Control

\[
\begin{align*}
Y &= \frac{KP + KDs}{s^2 + KDs + KP - 1} \\
\frac{Y}{R} &= \frac{KP + KDs}{s^2 + KDs + KP - 1}
\end{align*}
\]

By choosing \( KP, KD \), we can achieve arbitrary pole placement!!

Also note that the addition of P-gain moves the zero:

\[
KD + K_P = 0 \quad \text{LHP zero at } -\frac{K_P}{K_D}
\]

But what’s missing? \( \text{DC gain } = \left. \frac{Y}{R} \right|_{s=0} = \frac{K_P}{K_P - 1} \neq 1 \)

— can’t have perfect tracking of constant reference.
Proportional-Integral-Derivative (PID) Control

Let us try

\[ U = \left( K_P + K_D s + \frac{K_I}{s} \right) E \] – the classic three-term controller

In fact, let’s also throw in a constant disturbance:

\[ \frac{1}{s^2 - 1} \]

We will see that, with PID control, the goals of

- tracking a constant reference \( r \)
- rejecting a constant disturbance \( w \)

can be accomplished in one shot.
Proportional-Integral-Derivative (PID) Control

\[ Y = \frac{1}{s^2 - 1} (U + W), \quad U = \left( K_P + K_D s + \frac{K_I}{s} \right) (R - Y) \]

so \( Y = \frac{K_P + K_D s + \frac{K_I}{s}}{s^2 - 1} (R - Y) + \frac{1}{s^2 - 1} W \)

Simplify:

\[ (s^2 - 1)Y = \left( K_P + K_D s + \frac{K_I}{s} \right) (R - Y) + W \]

\[ \left( s^2 - 1 + K_P + K_D s + \frac{K_I}{s} \right) Y = \left( K_P + K_D s + \frac{K_I}{s} \right) R + W \]

\[ (s^3 - s + K_P s + K_D s^2 + K_I)Y = (K_P s + K_D s^2 + K_I)R + W s \]
Proportional-Integral-Derivative (PID) Control

\[ Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W \]
Proportional-Integral-Derivative (PID) Control

\[ Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R \]
\[ + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W \]

Stability:

- need \( K_D > 0, K_P > 1, K_I > 0 \) (necessary condition) and \( K_D(K_P - 1) > K_I \) (Routh–Hurwitz for 3rd-order)
- can still assign coefficients arbitrarily by choosing \( K_P, K_I, K_D \)
Proportional-Integral-Derivative (PID) Control

Reference tracking:

\[
Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} \bigg|_{s=0} = 1
\]

so, with the addition of I-feedback, we remove earlier limitation and achieve perfect tracking!
Proportional-Integral-Derivative (PID) Control

\[ Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R \]

\[ + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W \]

Disturbance rejection:

\[ \text{DC gain}(W \rightarrow Y) = \left. \frac{s}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 0 \]

— so, integral gain also gives complete attenuation of constant disturbances!!
When the actuator saturates, the error continues to be integrated, resulting in large overshoot.

We say that the integrator “winds up:” the error may be small, but its integrated past history builds up.

There are various anti-windup schemes to deal with this practically important issue. (Essentially, we attempt to detect the onset of saturation and turn the integrator off.)
The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.

Consider the reference $r(t) = \frac{t^k}{k!}1(t) \leftrightarrow R(s) = \frac{1}{s^{k+1}}$

Error signal: $E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$

FVT gives (assuming stability):

$$e(\infty) = sE(s) \bigg|_{s=0} = \frac{1}{1 + KP} \frac{1}{s^k} \bigg|_{s=0}$$

— let’s see how the forward gain affects tracking performance.
System type: the number $n$ of poles the forward-loop transfer function $KP$ has at the origin. It is the degree of the lowest-degree polynomial that cannot be tracked in feedback with zero steady-state error.

Note: the system type is calculated from the forward-loop transfer function, although the conclusions we will draw are about the closed-loop system.
Let's see how forward gain $KP$ affects tracking performance.

Let's suppose that $KP$ has $n$th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

$$sE(s) = \frac{1}{(1 + \frac{K_0}{s^n}) s^k} = \frac{s^{n-k}}{s^n + K_0}$$

— what about $sE(s) \bigg|_{s=0}$?
Let’s suppose that $KP$ has $n$th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

$$sE(s) = \frac{1}{(1 + \frac{K_0}{s^n}) s^k} = \frac{s^{n-k}}{s^n + K_0}$$

— what about $sE(s) \bigg|_{s=0}$?

Recall: reference $r(t)$ is a polynomial of degree $k$

Three cases to consider —

- $n > k$: $e(\infty) = 0$  
  perfect tracking
- $n = k$: $e(\infty) = \text{const} \neq 0$  
  imperfect tracking
- $n < k$: $e(\infty) = \infty$  
  no tracking
System type is the degree of the lowest-degree polynomial that cannot be tracked in feedback with zero steady-state error.

- **Type 0**: no pole at the origin. This is what we had without the I-gain: nonzero SS error to constant references.
- **Type 1**: a single pole at the origin. This is what we get with I-gain: can track (respectively, reject) constant references (respectively, disturbances) with zero error.
  - can check that we have a nonzero (but finite) error when tracking ramp references
- **Type 2**: a double pole at the origin. Can track ramp references without error, but not $t^2, t^3, ...$
**PID Control: Summary & Further Comments**

**P-gain** simplest to implement, but not always sufficient for stabilization

**D-gain** helps achieve stability, improves time response (more control over pole locations)

▶ arbitrary pole placement only valid for 2nd-order response; in general, we still have control over two *dominant poles*
▶ cannot be implemented directly, so need approximate implementation; D-gain also amplifies noise

**I-gain** essential for perfect steady-state tracking of constant reference and rejection of constant disturbance

▶ but $1/s$ is not a stable element by itself, so one must be careful: it can destabilize the system if the feedback loop is broken (*integrator wind-up*)