Plan of the Lecture

- Review: prototype 2nd-order system
- Today’s topic: transient response specifications

*Goal*: develop formulas and intuition for various features of the transient response: rise time, overshoot, settling time.

*Reading*: FPE, Sections 3.3–3.4; lab manual
Prototype 2nd-Order System

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

By the quadratic formula, the poles are:

\[
s = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}
\]

\[
= -\omega_n \left( \zeta \pm \sqrt{\zeta^2 - 1} \right)
\]

The nature of the poles changes depending on \( \zeta \):

- \( \zeta > 1 \) both poles are real and negative
- \( \zeta = 1 \) one negative pole
- \( \zeta < 1 \) two complex poles with negative real parts

\[ s = -\sigma \pm j\omega_d \]

where \( \sigma = \zeta\omega_n, \omega_d = \omega_n\sqrt{1 - \zeta^2} \)
Prototype 2nd-Order System

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta < 1 \]

The poles are

\[ s = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2} = -\sigma \pm j\omega_d \]

Note that

\[ \sigma^2 + \omega_d^2 = \zeta^2 \omega_n^2 + \omega_n^2 - \zeta^2 \omega_n^2 = \omega_n^2 \]

\[ \cos \phi = \frac{\zeta\omega_n}{\omega_n} = \zeta \]
2nd-Order Response

Let's compute the system's impulse and step response:

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2} \]

- **Impulse response:**

\[ h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{ \frac{\omega_n^2/\omega_d}{(s + \sigma)^2 + \omega_d^2} \right\} \]

\[ = \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t) \quad \text{(table, \#20)} \]

- **Step response:**

\[ \mathcal{L}^{-1}\left\{ \frac{H(s)}{s} \right\} = \mathcal{L}^{-1}\left\{ \frac{\sigma^2 + \omega_d^2}{s[(s + \sigma)^2 + \omega_d^2]} \right\} \]

\[ = 1 - e^{-\sigma t} \left( \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right) \quad \text{(table, \#21)} \]
2nd-Order Step Response

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2} \]

\[ u(t) = 1(t) \quad \rightarrow \quad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t)\right) \]

where \( \sigma = \zeta \omega_n \) and \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \) (damped frequency)

The parameter \( \zeta \) is called the *damping ratio*

- \( \zeta > 1 \): system is overdamped
- \( \zeta < 1 \): system is underdamped
- \( \zeta = 0 \): no damping
  \( (\omega_d = \omega_n) \)
2nd-Order Step Response

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2} \]

\[ u(t) = 1(t) \quad \rightarrow \quad y(t) = 1 - e^{-\sigma t} \left( \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right) \]

where \( \sigma = \zeta \omega_n \) and \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \) (damped frequency)

We will see that the parameters \( \zeta \) and \( \omega_n \) determine certain important features of the transient part of the above step response.

We will also learn how to pick \( \zeta \) and \( \omega_n \) in order to shape these features according to given specifications.
Transient Response Specifications: Rise Time

Let’s first take a look at 1st-order step response

\[ H(s) = \frac{a}{s + a}, \quad a > 0 \quad \text{(stable pole)} \]

DC gain = 1 (by FVT)

**Step response:**

\[ Y(s) = \frac{H(s)}{s} = \frac{a}{s(s + a)} = \frac{1}{s} - \frac{1}{s + a} \]

\[ y(t) = \mathcal{L}^{-1}\{Y(s)\} = 1(t) - e^{-at} \]

Rise time \( t_r \): the time it takes to get from 10% of steady-state value to 90%
Rise Time

Step response: \( y(t) = 1(t) - e^{-at} \)

Rise time \( t_r \): the time it takes to get from 10\% of steady-state value to 90\%

In this example, it is easy to compute \( t_r \) analytically:

\[
1 - e^{-at_{0.1}} = 0.1 \quad e^{-at_{0.1}} = 0.9 \quad t_{0.1} = -\frac{\ln 0.9}{a} \\
1 - e^{-at_{0.9}} = 0.9 \quad e^{-at_{0.9}} = 0.1 \quad t_{0.9} = -\frac{\ln 0.1}{a} \\
t_r = t_{0.9} - t_{0.1} = \frac{\ln 0.9 - \ln 0.1}{a} = \frac{\ln 9}{a} \approx \frac{2.2}{a}
\]
Transient Response Specs

Now let’s consider the more interesting case: **2nd-order response**

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2} \]

where \( \sigma = \zeta \omega_n \) \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \) \( (\zeta < 1) \)

**Step response:**
\[ y(t) = 1 - e^{-\sigma t} \left( \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right) \]
Transient-Response Specs

Step response:  
\[ y(t) = 1 - e^{-\sigma t} \left( \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right) \]

- rise time \( t_r \) — time to get from 0.1\( y(\infty) \) to 0.9\( y(\infty) \)
- overshoot \( M_p \) and peak time \( t_p \)
- settling time \( t_s \) — first time for transients to decay to within a specified small percentage of \( y(\infty) \) and stay in that range (we will usually worry about 5% settling time)
Transient-Response (or Time-Domain) Specs

Do we want these quantities to be large or small?

- $t_r$ small
- $M_p$ small
- $t_p$ small
- $t_s$ small

Trade-offs among specs: decrease $t_r$ $\rightarrow$ increase $M_p$, etc.
Rise time $t_r$ — hard to calculate analytically. Empirically, on the normalized time scale ($t \rightarrow \omega_n t$), rise times are approximately the same

$$w_n t_r \approx 1.8 \quad \text{(exact for } \zeta = 0.5)$$

So, we will work with $t_r \approx \frac{1.8}{\omega_n} \quad \text{(good approx. when } \zeta \approx 0.5)$
Formulas for TD Specs: Overshoot & Peak Time

$t_p$ is the first time $t > 0$ when $y'(t) = 0$

\[ y(t) = 1 - e^{-\sigma t} \left( \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right) \]

\[ y'(t) = \left( \frac{\sigma^2}{\omega_d} + \omega_d \right) e^{-\sigma t} \sin(\omega_d t) = 0 \text{ when } \omega_d t = 0, \pi, 2\pi, \ldots \]

so \( t_p = \frac{\pi}{\omega_d} \)
We have just computed $t_p = \frac{\pi}{\omega_d}$

To find $M_p$, plug this value into $y(t)$:

$$M_p = y(t_p) - 1 = -e^{-\frac{\sigma \pi}{\omega_d}} \left( \cos \left( \omega_d \frac{\pi}{\omega_d} \right) + \frac{\sigma}{\omega_d} \sin \left( \omega_d \frac{\pi}{\omega_d} \right) \right)$$

$$= \exp \left( -\frac{\sigma \pi}{\omega_d} \right) = \exp \left( -\frac{\pi \zeta}{\sqrt{1-\zeta^2}} \right) \quad \text{— exact formula}$$
Formulas for TD Specs: Settling Time

\[ t_s = \min \left\{ t > 0 : \frac{|y(t') - y(\infty)|}{y(\infty)} \leq 0.05 \text{ for all } t' \geq t \right\} \] (here, \( y(\infty) = 1 \))

\[ |y(t) - 1| = e^{-\sigma t} \left| \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right| \]

here, \( e^{-\sigma t} \) is what matters (\( \sin \) and \( \cos \) are bounded between \( \pm 1 \)), so \( e^{-\sigma t} \leq 0.05 \) this gives \( t_s = -\frac{\ln 0.05}{\sigma} \approx \frac{3}{\sigma} \)
Formulas for TD Specs

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s + \sigma)^2 + \omega_d^2} \]

\[ t_r \approx \frac{1.8}{\omega_n} \]
\[ t_p = \frac{\pi}{\omega_d} \]
\[ M_p = \exp \left( -\frac{\pi\zeta}{\sqrt{1 - \zeta^2}} \right) \]
\[ t_s \approx \frac{3}{\sigma} \]
TD Specs in Frequency Domain

We want to visualize time-domain specs in terms of admissible pole locations for the 2nd-order system

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s + \sigma)^2 + \omega_d^2} \]

where \( \sigma = \zeta \omega_n \)

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]

Step response: \( y(t) = 1 - e^{-\sigma t} \left( \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right) \)
Rise Time in Frequency Domain

Suppose we want $t_r \leq c$ \hspace{1em} (c is some desired given value)

\[ t_r \approx \frac{1.8}{\omega_n} \leq c \quad \implies \quad \omega_n \geq \frac{1.8}{c} \]

Geometrically, we want poles to lie in the shaded region:

(recall that $\omega_n$ is the magnitude of the poles)
Overshoot in Frequency Domain

Suppose we want $M_p \leq c$

$$M_p = \exp \left( -\frac{\pi \zeta}{\sqrt{1 - \zeta^2}} \right) \leq c \quad \text{— need large damping ratio}$$

Geometrically, we want poles to lie in the shaded region:

\[ \frac{\dot{\zeta}}{\sqrt{1 - \zeta^2}} = \frac{\omega_n \zeta}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\sigma}{\omega_d} = \cot \varphi \]

\[ \text{— need } \varphi \text{ to be small} \]

Intuition: good damping $\rightarrow$ good decay in $1/2$ period
Settling Time in Frequency Domain

Suppose we want $t_s \leq c$

$$t_s \approx \frac{3}{\sigma} \leq c \quad \implies \quad \sigma \geq \frac{3}{c}$$

Want poles to be sufficiently fast (large enough magnitude of real part):

Intuition: poles far to the left $\rightarrow$ transients decay faster $\rightarrow$ smaller $t_s$
Combination of Specs

If we have specs for any combination of $t_r, M_p, t_s$, we can easily relate them to allowed pole locations:

The shape and size of the region for admissible pole locations will change depending on which specs are more severely constrained.

This is very appealing to engineers: easy to visualize things, no such crisp visualization in time domain.

But: not very rigorous, and also only valid for our prototype 2nd-order system, which has only 2 poles and no zeros ...