Plan of the Lecture

- **Review:** transient and steady-state response; DC gain and the FVT
- **Today’s topic:** system-modeling diagrams; prototype 2nd-order system

*Goal*: develop a methodology for representing and analyzing systems by means of block diagrams; start analyzing a prototype 2nd-order system.

*Reading*: FPE, Sections 3.1–3.2; lab manual
System Modeling Diagrams

large system $\xleftarrow{\text{decompose}} \xrightarrow{\text{compose}}$ smaller blocks (subsystems)

— this is the core of systems theory

We will take smaller blocks from some given library and play with them to create/build more complicated systems.
All-Integrator Diagrams

Our library will consist of three building blocks:

\[ \dot{y} \xrightarrow{1/s} y \quad (or \; sY) \quad (or \; Y) \]

integrator

\[ u_1 + u_2 = y \]

summing junction

\[ u \xrightarrow{a} y = au \]

constant gain

Two warnings:

- We can (and will) work either with \( u, y \) (time domain) or with \( U, Y \) (s-domain) — will often go back and forth
- When working with block diagrams, we typically ignore initial conditions.

This is the \textit{lowest level} we will go to in lectures; in the labs, you will implement these blocks using op amps.
Example 1

Build an all-integrator diagram for

\[ \ddot{y} = u \iff s^2 Y = U \]

This is obvious:

\[ u \rightarrow \frac{1}{s} \rightarrow \dot{y} \rightarrow \frac{1}{s} \rightarrow y \]

or

\[ U \rightarrow \frac{1}{s} \rightarrow sY \rightarrow \frac{1}{s} \rightarrow Y \]
Example 2
(building on Example 1)

\[
\ddot{y} + a_1 \dot{y} + a_0 y = u \quad \iff \quad s^2 Y + a_1 s Y + a_0 Y = U
\]

or

\[
Y(s) = \frac{U(s)}{s^2 + a_1 s + a_0}
\]

Always solve for the highest derivative:

\[
\ddot{y} = -a_1 \dot{y} - a_0 y + u \quad \Longrightarrow \quad = v
\]
Example 3

Build an all-integrator diagram for a system with transfer function

\[ H(s) = \frac{b_1s + b_0}{s^2 + a_1s + a_0} \]

Step 1: decompose \( H(s) = \frac{1}{s^2 + a_1s + a_0} \cdot (b_1s + b_0) \)

\[ U \xrightarrow{\frac{1}{s^2 + a_1s + a_0}} X \xrightarrow{b_1s + b_0} Y \]

— here, \( X \) is an auxiliary (or intermediate) signal

Note: \( b_0 + b_1s \) involves differentiation, which we cannot implement using an all-integrator diagram. But we will see that we don’t need to do it directly.
Example 3, continued

Step 1: decompose $H(s) = \frac{1}{s^2 + a_1 s + a_0} \cdot (b_1 s + b_0)$

\[
\begin{align*}
U & \quad \frac{1}{s^2 + a_1 s + a_0} \quad X \\
& \quad b_1 s + b_0 \quad Y
\end{align*}
\]

Step 2: The transformation $U \rightarrow X$ is from Example 2:

\[
\begin{align*}
U & \quad + s^2 X \\
& \quad \frac{1}{s} \quad sX \quad \frac{1}{s} \quad X
\end{align*}
\]
Example 3, continued

Step 3: now we notice that

\[ Y(s) = b_1 sX(s) + b_0 X(s), \]

and both \( X \) and \( sX \) are available signals in our diagram. So:

![Diagram showing the flow of signals with labels for \( U, s^2X, sX, X, b_1, b_0, a_1, a_0 \), and the output \( Y \).]
Example 3, continued

All-integrator diagram for $H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$

Can we write down a state-space model corresponding to this diagram?
Example 3, continued

State-space model:

\[ s^2 X = U - a_1 sX - a_0 X \]
\[ \ddot{x} = -a_1 \dot{x} - a_0 x + u \]
\[ Y = b_1 sX + b_0 X \]
\[ y = b_1 \dot{x} + b_0 x \]
Example 3, continued

State-space model:

\[
\dot{x} = -a_1 \dot{x} - a_0 x + u \quad y = b_1 \dot{x} + b_0 x
\]

\[
x_1 = x, \quad x_2 = \dot{x}
\]

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-a_0 & -a_1
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
0 \\
1
\end{pmatrix} u \quad y = \begin{pmatrix}
b_0 & b_1
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

This is called *controller canonical form*.

- Easily generalizes to dimension > 1
- The reason behind the name will be made clear later in the semester
All-integrator diagram for $H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$

State-space model:

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-a_0 & -a_1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u
\quad y = (b_0 \quad b_1)
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
$$

**Important**: for a given $H(s)$, the diagram is *not unique*. But, once we build a diagram, the state-space equations are unique (up to coordinate transformations).
Basic System Interconnections

Now we will take this a level higher — we will talk about building complex systems from smaller blocks, without worrying about how those blocks look on the inside (they could themselves be all-integrator diagrams, etc.)

Block diagrams are an *abstraction* (they hide unnecessary “low-level” detail ...)

Block diagrams describe the *flow of information*
Basic System Interconnections: Series & Parallel

Series connection

\[ U \rightarrow G_1 \rightarrow G_2 \rightarrow Y \]

\[
\frac{Y}{U} = G_1 G_2
\]

(G is common notation for t.f.’s)

Parallel connection

\[ U \rightarrow G_1 \rightarrow + \rightarrow G_2 \rightarrow + \rightarrow Y \]

\[
\frac{Y}{U} = G_1 + G_2
\]

(for SISO systems, the order of \( G_1 \) and \( G_2 \) does not matter)
Find the transfer function from $R$ (reference) to $Y$

\[
Y = \frac{G_1}{1 + G_1 G_2} R
\]
Basic System Interconnections: Negative Feedback

The gain of a negative feedback loop:

\[
Y = \frac{G_1}{1 + G_1 G_2} R
\]

This is an important relationship, easy to derive — no need to memorize it.
Unity Feedback

Other feedback configurations are also possible:

This is called *unity feedback* — no component on the feedback path.

Common structure (saw this in Lecture 1):

- $R = \text{reference}$
- $U = \text{control input}$
- $Y = \text{output}$
- $E = \text{error}$
- $G_1 = \text{plant (also denoted by } P \text{)}$
- $G_2 = \text{controller or compensator (also denoted by } C \text{ or } K \text{)}$
Let’s practice with deriving transfer functions:

- **Reference** \( R \) to output \( Y \):
  \[
  \frac{Y}{R} = \frac{G_1 G_2}{1 + G_1 G_2}
  \]

- **Reference** \( R \) to control input \( U \):
  \[
  \frac{U}{R} = \frac{G_2}{1 + G_1 G_2}
  \]

- **Error** \( E \) to output \( Y \):
  \[
  \frac{Y}{E} = G_1 G_2 \quad \text{(no feedback path)}
  \]
Block Diagram Reduction

Given a complicated diagram involving series, parallel, and feedback interconnections, we often want to write down an overall transfer function from one of the variables to another. This requires lots of practice: read FPE, Section 3.2 for examples.

General strategy:

- Name all the variables in the diagram
- Write down as many relationships between these variables as you can
- Learn to recognize series, parallel, and feedback interconnections
- Replace them by their equivalents
- Repeat
Prototype 2nd-Order System

So far, we have only seen transfer functions that have either real poles or purely imaginary poles:

\[
\frac{1}{s + a'}, \quad \frac{1}{(s + a)(s + b)'} \quad \frac{1}{s^2 + \omega^2}
\]

We also need to consider the case of complex poles, i.e., ones that have \(\text{Re}(s) \neq 0\) and \(\text{Im}(s) \neq 0\).

For now, we will only look at second-order systems, but this will be sufficient to develop some nontrivial intuition (dominant poles).

Plus, you will need this for Lab 1.
Prototype 2nd-Order System

Consider the following transfer function:

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

Comments:

- $\zeta > 0, \omega_n > 0$ are arbitrary parameters
- the denominator is a general 2nd-degree monic polynomial, just written in a weird way
- $H(s)$ is normalized to have DC gain $= 1$ (provided DC gain exists)
Prototype 2nd-Order System

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

By the quadratic formula, the poles are:

\[
    s = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \\
    = -\omega_n \left( \zeta \pm \sqrt{\zeta^2 - 1} \right)
\]

The nature of the poles changes depending on \( \zeta \):

- \( \zeta > 1 \) both poles are real and negative
- \( \zeta = 1 \) one negative pole
- \( \zeta < 1 \) two complex poles with negative real parts

\[
    s = -\sigma \pm j\omega_d \\
    \text{where} \quad \sigma = \zeta\omega_n, \quad \omega_d = \omega_n \sqrt{1 - \zeta^2}
\]
Prototype 2nd-Order System

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta < 1 \]

The poles are

\[ s = -\zeta \omega_n \pm j\omega_n \sqrt{1 - \zeta^2} = -\sigma \pm j\omega_d \]

Note that

\[ \sigma^2 + \omega_d^2 = \zeta^2 \omega_n^2 + \omega_n^2 - \zeta^2 \omega_n^2 = \omega_n^2 \]

\[ \cos \varphi = \frac{\zeta \omega_n}{\omega_n} = \zeta \]
2nd-Order Response

Let’s compute the system’s impulse and step response:

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2} \]

- **Impulse response:**

\[
h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{ \frac{(\omega_n^2/\omega_d)\omega_d}{(s + \sigma)^2 + \omega_d^2} \right\} = \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t) \quad \text{(table, \#20)}
\]

- **Step response:**

\[
\mathcal{L}^{-1}\left\{ \frac{H(s)}{s} \right\} = \mathcal{L}^{-1}\left\{ \frac{\sigma^2 + \omega_d^2}{s[(s + \sigma)^2 + \omega_d^2]} \right\} = 1 - e^{-\sigma t} \left( \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right) \quad \text{(table, \#21)}
\]
2nd-Order Step Response

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2} \]

\[ u(t) = 1(t) \quad \rightarrow \quad y(t) = 1 - e^{-\sigma t} \left( \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right) \]

where \( \sigma = \zeta \omega_n \) and \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \) (damped frequency)

The parameter \( \zeta \) is called the damping ratio

- \( \zeta > 1 \): system is overdamped
- \( \zeta < 1 \): system is underdamped
- \( \zeta = 0 \): no damping \( (\omega_d = \omega_n) \)