Overview of state space design:

The goal of any design method is to determine the control signal \( u \) that causes the output \( y \) to meet certain design specs. The state space design method often proceeds as follows:

1) Translate the design specs into a set of desired closed-loop pole locations

2) Find a state space realization \((A, B, C, D)\) of the plant \( G_p(s) \), i.e.

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

where \( C (sI-A)^{-1}B = G_p(s) \)
A linear state feedback control is then of the form

\[ U = -k_1 x_1 - k_2 x_2 - \cdots - k_n x_n \]

\[ = -Kx \quad K = \begin{bmatrix} k_1, \ldots, k_n \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \]

3) Find \( K \) so that the matrix \( \tilde{A} = A - BK \) has eigenvalues at the desired closed loop poles.

Most often, the output \( y \) can be directly measured but not the states \( x_1, \ldots, x_n \) as this would require \( n \) sensors, one for each \( x_i \).

So the control \( U = -Kx \) cannot be implemented. A state estimator is a system with inputs \( u \) and \( y \) and output \( \hat{x} \). The goal of a state estimator is

\[ \lim_{t \to \infty} (x(t) - \hat{x}(t)) = 0. \]

\[ y \quad \rightarrow \quad \text{estimator} \quad \rightarrow \quad \hat{x}_1 \ldots \hat{x}_n \]

\[ y \quad \rightarrow \quad \hat{x}_1 \ldots \hat{x}_n \]
Certainty Equivalence

The principle of 'certainty-equivalence' states that the control law

\[ u = -K\hat{x} \]

i.e. using \( \hat{x} \) in place of the true state \( x \), still satisfies the design specifications. We will make this precise and also detail ways to design both the controller and the estimator.

2.5.1 Finding the control law

- Pole assignment

In very simple cases one can find the controller gains \( k_1, \ldots, k_n \) by 'matching coefficients'. We illustrate this with an example.

Ex: pendulum (linearized)

\[ \ddot{\theta} + \omega^2 \theta = u \Rightarrow G_p(s) = \frac{1}{s^2 + \omega^2} \]
with $x_1 = \Theta$, $x_2 = \dot{\Theta}$ the state equations are

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-w^2 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u
$$

Suppose we want the closed loop system to have damping ratio $\delta = 1$ and natural frequency $2\omega^2$. Thus we want the closed loop characteristic polynomial to be $s^2 + 4\omega^2 s + 4\omega^2$

The control $u = -k_1 x_1 - k_2 x_2 = -k x$

results in

$$
\dot{x} = A x + B u = (A - BK) x
$$

$$
\tilde{A} = A - BK =
\begin{bmatrix}
0 & 1 \\
-w^2 & 0
\end{bmatrix} -
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
k_1 \\
k_2
\end{bmatrix}
$$

$$
= 
\begin{bmatrix}
0 & 1 \\
-w^2 & 0
\end{bmatrix} -
\begin{bmatrix}
0 & 0 \\
k_1 & k_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-w^2 & -k_1 - k_2
\end{bmatrix}
$$

$$
\det(sI - \tilde{A}) = 
\det
\begin{bmatrix}
s & -1 \\
w^2 + k_1 & s + k_2
\end{bmatrix} = s^2 + k_2 s + \omega^2 + k_1
$$

Equating coefficients gives $k_2 = 4\omega$ \quad $w^2 + k_1 = 4\omega^2$

$\Rightarrow$ \quad $k_1 = 3\omega^2$; \quad $k_2 = 4\omega$
Notice that the control law
\[ u = -k_1 x_1 - k_2 x_2 = -3 \omega^2 \theta - 4 \dot{\omega} \theta \]
is a PD-type control.

**Ackermann's Formula:**

Suppose that the state equations are in control canonical form

\[
\begin{bmatrix}
-a_1 & -a_2 & \cdots & -a_n \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & 0
\end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix}
b_1, b_2, \ldots, b_n
\end{bmatrix}
\]

representing the transfer function
\[
G(s) = \frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}
\]

Notice that the upper row is just the coefficients of the characteristic polynomial.

With
\[
u = -K x = -k_1 x_1 - \cdots - k_n x_n
\]
we have
\[
F_c - G_c K = \begin{bmatrix}
-a_1 - k_1 & -a_2 - k_2 & \cdots & -a_n - k_n \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & 0
\end{bmatrix}
\]
Therefore it is easy to see by inspection that if the desired closed loop characteristic polynomial is

$$\chi_c(s) = s^n + a_1 s^{n-1} + \ldots + a_n$$

then the control gains

$$k_1 = -a_1 + a_n, \quad k_2 = -a_2 + a_n, \ldots, \quad k_n = -a_n + a_n$$

results in

$$\det(s I - \chi_c G_c) = \chi_c(s).$$

Therefore, given an arbitrary controllable system

$$\dot{x} = FX + GU$$

and a desired closed loop char. poly. $\chi_c(s)$

1) Find $T$ such that

$$T^{-1}FT = F_c, \quad T^{-1}G = G_c$$

i.e. find $x = Tz$ so that

$$\dot{z} = F_c z + G_c U$$

2) Find $K_c$ by inspection so that

$$\det(s F_c + G_c K_c) = \chi_c(s)$$

3) In terms of $x$ then the control

$$U = -K_c z = -K_c T^{-1} x$$

i.e.

$$K = -K_c T^{-1}$$
Ackermann's formula consolidates this 3-step procedure as

$$K = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix} C^{-1} x_c(F)$$

where

$$C = \begin{bmatrix} G & FG & F^2G & \ldots & F^{n-1}G \end{bmatrix}$$

$$x_c(F) = F^n + x_1 F^{n-1} + \ldots + x_n I$$

Example: The linearized pendulum system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -w^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

with $$x_c(s) = s^2 + x_1 s + x_2 = s^2 + 4w^2 s + 4w^2$$

so:

$$x_c(F) = \begin{bmatrix} 0 & 1 \\ -w^2 & 0 \end{bmatrix}^2 + 4w \begin{bmatrix} 0 & 1 \\ -w^2 & 0 \end{bmatrix} + \begin{bmatrix} 4w^2 & 0 \\ 0 & 4w^2 \end{bmatrix} = \begin{bmatrix} 3w^2 & 4w \\ -4w^2 & 3w^2 \end{bmatrix}$$

The controllability matrix is

$$C = \begin{bmatrix} G & FG \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow C^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore Ackermann's formula gives

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3w^2 & 4w \\ -4w^2 & 3w^2 \end{bmatrix} = \begin{bmatrix} 3w^2 & 4w \\ -4w^2 & 3w^2 \end{bmatrix}$$

which is the same result achieved previously.
State Estimation

One way to estimate the state $x(t)$ is to create a model of the system

$$\dot{\hat{x}} = F \hat{x} + gu$$

where $\hat{x}(t)$ is the estimate of $x(t)$.

Q: Why won’t this work?
A: We don’t know the initial condition $x(0)$. If we did we could set $\hat{x}(0) = x(0)$ and the solution $\hat{x}(t)$ would equal $x(t)$ for all $t$. The best we can hope for is that

$$\hat{x}(t) = x(t) - x(t) \to 0 \text{ as } t \to \infty$$

$\hat{x}(t)$ is the estimation error. To achieve this we use feedback. We construct an observer, which is a dynamical system of the form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

$L$ is called the observer gain and, of course, $y = Cx$ is measurable.
so far we have

\[ \dot{x} = Ax + Bu \]
\[ \dot{\hat{x}} = A\hat{x} + Bu + L(y - Cx) \]

If we subtract the first two equations we get

\[ \dot{\hat{x}} = \dot{x} - \dot{\hat{x}} = A(x - \hat{x}) - LC(x - \hat{x}) \]

or

\[ \ddot{x} = (A - LC)\hat{x} \]

Note that if \( A - LC \) has all eigenvalues with negative real part then \( \hat{x} \to 0 \) for all initial conditions, i.e. \( \hat{x}(t) \to x(t) \) as \( t \to \infty \) (exponentially)

So the observer design problem reduced to, given \( (A, C) \) find \( L \) so that \( \bar{A} = A - LC \) is a Hurwitz matrix.

Example: Estimator design for the linearised pendulum
\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -w^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]
\[ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \]

i.e. \( x_1 = \theta \) is measured (say, by an optical encoder) but there is no measurement of \( x_2 = \dot{\theta} \).

Design an estimator to place the estimator poles at \(-10w\). The corresponding characteristic equation is:

\[ \chi_e(s) = (s + 10w)^2 = s^2 + 20ws + 100w^2 \]

with \( \chi = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \) we have \( 5I - (A - LC) \)

\[ = 5I - A + LC = \begin{bmatrix} s & -1 \\ w^2 & s \end{bmatrix} + \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ = \begin{bmatrix} s - 1 \\ w^2 s + \theta_2 \end{bmatrix} + \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} s + \theta_1 - 1 \\ w^2 s + \theta_2 \end{bmatrix} \]

\[ \text{det}(5I - A + LC) = s^2 + \theta_1 s + w^2 + \theta_2^2 = s^2 + 20ws + 100w^2 \]

\[ \Rightarrow \theta_1 = 20w; \quad \theta_2 = 99w^2 \]

What happens when we use the previously designed control law \( u = -Kx \) using the estimated state \( \hat{x} \) in place of \( x \)?
The complete system is:
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
\dot{u} &= -Kx \\
\hat{x} &= A\hat{x} + Bu + L(y - C\hat{x}) \\
y &= Cx
\end{align*}
\]

We can rewrite this system as:
\[
\begin{align*}
\dot{x} &= Ax - BK\hat{x} = A\hat{x} - BK[\hat{x} + x - x] \\
&= (A - BK)x + BK\tilde{x} \\
\hat{x} &= (A - LC)\tilde{x}
\end{align*}
\]

Or,
\[
\begin{bmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{bmatrix} =
\begin{bmatrix}
A - BK & BK \\
0 & A - LC
\end{bmatrix}
\begin{bmatrix}
x \\
\hat{x}
\end{bmatrix} = A' \begin{bmatrix}
x \\
\hat{x}
\end{bmatrix}
\]

Therefore, the determinant of \( A' \) is
\[
\det \begin{bmatrix}
A - BK & BK \\
0 & A - LC
\end{bmatrix} = \det(SI - A + BK) \det(SI - A + LC)
\]

Therefore, the poles of the closed loop system are the union of eigenvalues of
\[
\begin{align*}
A - BK & \text{ (controller poles)} \\
A - LC & \text{ (observer poles)}
\end{align*}
\]
as a rule of thumb we should place the observer poles far to the left so that $\hat{x} \approx x$ fast; after a short time $\hat{x} \approx x$ and so the control $u = -k \hat{x} \approx -k x$.

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The limiting factor on how far to the left the observer poles can be placed is sensor noise.

The limiting factor on how far to the left the controller poles can be placed is actuator saturation.
Observer canonical form:

The observer canonical form is another state space realization for a transfer function

\[ G_p(s) = \frac{b_1 s^{n-1} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n} \]

It takes the form

\[ \dot{x} = F_0 x + G_0 u \]
\[ y = H_0 x \]

where

\[ F_0 = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ -a_n & 0 & \cdots & 0 & 1 \\ -a_n & 0 & \cdots & 0 & 0 \end{bmatrix} \]

\[ G_0 = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad H_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \]

A block diagram of this realization is given below.
notice that the output is feedback directly to each integrator.

One of the advantages of the observer canonical form is that the observer gain $L$ can be found by inspection. With $L = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_n \end{bmatrix}$ we have

$$F_0 - LH_0 = \begin{bmatrix} -a_1 - \ell_1 & 1 & 0 & \cdots & 0 \\ -a_2 - \ell_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n - \ell_n & 0 & \cdots & 0 & 1 \end{bmatrix}$$

which has characteristic equation

$$s^n + (a_1 + \ell_1)s^{n-1} + \cdots + a_n + \ell_n, \text{ so } 0$$
achieve a characteristic polynomial

\[ s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n \]

for the estimator error system, we need only choose

\[ l_i = \alpha_i - a_i, \quad \ldots \quad l_n = \alpha_n - a_n \]

As in the control canonical form case, we can transform a given system to control observer canonical form provided the system is observable, which means

\[
\Theta = \begin{bmatrix}
H \\
HF \\
HF^2 \\
\vdots \\
HF^{n-1}
\end{bmatrix}
\]

is non-singular.

Ackermann's formula to compute the observer gains is given by

\[
L = \chi_e(t) \Theta^{-1} \begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

where \( \chi_e(s) \) is the desired characteristic polynomial for the estimation error system.
**Duality**

Duality refers to a mathematical equivalence between the control and estimation problems as follows: Consider systems 1 and 2, where

1. \( (A, B, C) \) : \( x = Ax + Bu, \ y = Cx \)
2. \( (A^T, C^T, B^T) \) : \( x = A^T x + C^T u, \ y = B^T x \)

Then, if \( 1 \) is controllable, \( 2 \) is observable, etc. since

\[
\begin{bmatrix}
B, AB, A^2B, \ldots, A^{n-1}B
\end{bmatrix}^T =
\begin{bmatrix}
B^T \\
B^T A^T \\
\vdots \\
B^T A^{n-1}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
C \\
C^2 \\
\vdots \\
C^{n-1}
\end{bmatrix}^T =
\begin{bmatrix}
C^T, A^T C^T, \ldots, A^{n-1} C^T
\end{bmatrix}
\]

This property allows the same algorithms to be used for placing control and estimator poles.