Reading: FPE, Section 7.4.1

Suppose we have a linear system with transfer function $\mathbf{H}(s)$ (which can be a matrix, in general). We have seen that the transfer function is related to the matrices in the state space model via $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$. In this section, we examine the question: How many choices of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ produce this transfer function $\mathbf{H}(s)$? We will see that there are multiple models (or **realizations**) that correspond to the same transfer function.

1 Controllable Canonical Form

Example. Consider the system $y^{(3)} + 7\ddot{y} + 14\dot{y} + 8y = \ddot{u} - 2\dot{u} + 3u$. Draw an all-integrator block diagram for this system, and create the corresponding state-space model.
Solution.

To generalize the above model, consider the transfer function

$$H(s) = \frac{a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \quad .$$

Note that the above form also captures transfer functions that have numerator polynomials with degree less than $n-1$ by setting the appropriate coefficients a_i to zero. By using the same technique as in the example above, an all-integrator block diagram for this transfer function is given by:

Based on this diagram, the state-variables evolve according to

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -b_0 x_1 - b_1 x_2 - \cdots - b_{n-2}x_{n-1} - b_{n-1}x_n + u \\
y &= a_0 x_1 + a_1 x_2 + \cdots + a_{n-2}x_{n-1} + a_{n-1}x_n \quad .
\end{aligned}$$

In matrix-vector form, this corresponds to:

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix} \mathbf{x} \quad .
\end{aligned}$$

This form is called the **controllable canonical form** (for reasons that we will see later). Note how the coefficients of the transfer function show up in the matrix: each of the denominator coefficients shows up negated and in reverse order in the bottom row of $\mathbf{A}$. The matrix $\mathbf{B}$ consists of all zeros, except for the last entry, which is 1. The $\mathbf{C}$ matrix has the numerator coefficients in reverse order. The most important point is that the **coefficients of the denominator polynomial are readily available in this form**.

2 Modal Canonical Form

We will now develop another commonly used state-space realization.

Example. Consider again the system $y^{(3)}+7\ddot{y}+14\dot{y}+8y = \ddot{u}-2\dot{u}+3u$. Find the partial fraction expansion of the transfer function $H(s) = \frac{Y(s)}{U(s)}$, and then draw an all-integrator block diagram by linking together block diagrams for each of the terms in the expansion. Use this block diagram to create a state-space model.

Solution.

To generalize this realization, consider the transfer function

$$H(s) = \frac{a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \ .$$

Suppose that the transfer function $H(s)$ has n distinct poles $-p_1, -p_2, \ldots, -p_n$, and consider the partial fraction expansion of $H(s)$:

$$H(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n} \ ,$$

where each of the k_i's are some constants. By using the same technique as in the above example, the all-integrator block diagram for this expansion is given by:

The state variables for this diagram evolve according to

$$\begin{aligned}
\dot{x}_1 &= -p_1 x_1 + k_1 u \\
\dot{x}_2 &= -p_2 x_2 + k_2 u \\
&\vdots \\
\dot{x}_n &= -p_n x_n + k_n u \\
y &= x_1 + x_2 + \cdots + x_n \ .
\end{aligned}$$

In matrix-vector form, this corresponds to:

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} -p_1 & 0 & 0 & \cdots & 0 \\ 0 & -p_2 & 0 & \cdots & 0 \\ 0 & 0 & -p_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -p_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_n \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \mathbf{x} \ .
\end{aligned}$$

Note that in this realization, the matrix $\mathbf{A}$ is a **diagonal** matrix, with the poles on the diagonal. In the state-space context, the poles of a linear system are also sometimes called the **modes** of the system, and so this realization is called the **modal** form of the system. Note that the $\mathbf{B}$ matrix has the coefficients of the partial fraction expansion as its entries, and the $\mathbf{C}$ matrix consists of all 1's (this is actually not unique, as we can also come up with a modal realization where $\mathbf{B}$ has all ones, and $\mathbf{C}$ has the coefficients of the expansion).

Note: The eigenvalues of a diagonal matrix (or, more generally, an upper or lower triangular matrix) are simply the elements on the diagonal of the matrix. Thus, the modal form corroborates the fact that the eigenvalues of $\mathbf{A}$ are the poles of the system.

3 Similarity Transformations

The above development shows that a given transfer function $\mathbf{H}(s)$ does not have a unique state-space realization (there are at least two: the controllable canonical form and the modal canonical form). Exactly how many realizations are there?

To answer this question, suppose we consider any particular realization

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}$$

for the transfer function $\mathbf{H}(s)$ (so that $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$). Now, let us choose an arbitrary invertible $n \times n$ matrix $\mathbf{T}$, and define a new state vector

$$\hat{\mathbf{x}} = \mathbf{T}\mathbf{x}\ .$$

In other words, the states in the vector $\hat{\mathbf{x}}$ are simply linear combinations of the states in the vector $\mathbf{x}$. Since $\mathbf{T}$ is a constant matrix, we have

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{T}\dot{\mathbf{x}} = \mathbf{T}\mathbf{A}\mathbf{x} + \mathbf{T}\mathbf{B}\mathbf{u} = \underbrace{\mathbf{T}\mathbf{A}\mathbf{T}^{-1}}_{\hat{\mathbf{A}}}\hat{\mathbf{x}} + \underbrace{\mathbf{T}\mathbf{B}}_{\hat{\mathbf{B}}}\mathbf{u}\\ \mathbf{y} &= \mathbf{C}\mathbf{x} = \underbrace{\mathbf{C}\mathbf{T}^{-1}}_{\hat{\mathbf{C}}}\hat{\mathbf{x}}\ .\end{aligned}$$

Thus, after this transformation, we obtain the new state-space model

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u}\\ \mathbf{y} &= \hat{\mathbf{C}}\hat{\mathbf{x}}\ .\end{aligned}$$

The transfer function corresponding to this model is given by

$$\begin{aligned}\hat{\mathbf{H}}(s) = \hat{\mathbf{C}}(s\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} &= \mathbf{C}\mathbf{T}^{-1}(s\mathbf{I} - \mathbf{T}\mathbf{A}\mathbf{T}^{-1})^{-1}\mathbf{T}\mathbf{B}\\ &= \mathbf{C}\mathbf{T}^{-1}(s\mathbf{T}\mathbf{T}^{\mathbf{-1}} - \mathbf{T}\mathbf{A}\mathbf{T}^{-1})^{-1}\mathbf{T}\mathbf{B}\\ &= \mathbf{C}\mathbf{T}^{-1}\mathbf{T}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{T}^{-1}\mathbf{T}\mathbf{B}\\ &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\\ &= \mathbf{H}(s)\ .\end{aligned}$$

Thus the transfer function for the realization with state-vector $\hat{\mathbf{x}}$ is the same as the transfer function for the realization with state-vector $\mathbf{x}$. For this reason, the transformation $\hat{\mathbf{x}} = \mathbf{T}\mathbf{x}$ is called a **similarity**

transformation. Since $\mathbf{T}$ can be *any* invertible matrix, and since there are an infinite number of invertible $n \times n$ matrices to choose from, we see that there are an **infinite number of realizations** for any given transfer function $\mathbf{H}(s)$.

Note: Since both of the sets $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ produce the same transfer function $\mathbf{H}(s)$, and since the poles of $\mathbf{H}(s)$ are the eigenvalues of $\mathbf{A}$ and also $\hat{\mathbf{A}}$, we see that the following relationship holds:

> The eigenvalues of $\mathbf{A}$ are the same as the eigenvalues of $\mathbf{TAT}^{-1}$ for any $n \times n$ invertible matrix $\mathbf{T}$.

This is a well known result in matrix theory.

Example. Consider the state-space model

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \ .
\end{aligned}
$$

Perform a similarity transformation with matrix $\mathbf{T} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and verify that the new realization produces the same transfer function as the above system.

Solution.