We have seen so far that the phase margin of a given system is representative of the system’s stability, and is directly related to the damping of the system – a larger phase margin makes the system more stable, and increases the damping. Given a system, we thus want to design a controller that improves the phase margin. In certain systems, one way to do this would be to decrease the gain of the system, so that the gain crossover frequency moves to the left (in the direction of increasing phase). However, we have seen that the low frequency gain of the system is related to how well the system tracks reference inputs – a larger low frequency gain corresponds to better tracking. Another metric is the gain crossover frequency: since the gain crossover frequency is approximately equal to the bandwidth and the natural frequency of the system, a larger gain crossover frequency corresponds to faster response, but also leads to smaller phase margins. Therefore, we would like to design more sophisticated controllers in order to keep the low frequency gain large (in order to meet tracking specifications), or to increase the gain crossover frequency (in order to obtain faster transients), and also to increase the phase at the gain crossover frequency (in order to boost the phase margin).

Recall from earlier parts of the course that PD and lead compensators are useful for improving stability, since the zero of the compensator tends to pull the root locus into the left half plane. We can therefore expect that lead compensators will be useful for improving the phase margin. We have seen how to design lead compensators using the root locus method, and now we will see how to design them using a purely Bode plot approach.

First, consider the standard unity feedback loop:

$$C(s) = K_c \frac{s + \alpha p}{s + p},$$

where $p > z$. Since $p > z$, we can write $z = \alpha p$ for some $0 < \alpha < 1$. The Bode form of the above controller is then given by

$$C(s) = K_c \frac{s + \alpha p}{s + p} = K_c \frac{\alpha \left( \frac{s}{\alpha p} + 1 \right)}{\left( \frac{s}{p} + 1 \right)} = \frac{s}{p + 1}.$$

The phase margin of the closed loop system can be obtained by examining the Bode plot of $C(s)P(s)$, which is obtained by simply adding together the Bode plots of $C_l(s)$ and $KP(s)$ (since the magnitude is on a log scale, and the phases inherently add). The gain $K$ of the compensator can be first be chosen to meet steady state error specifications, or to obtain a certain crossover frequency. Once that is done, let’s
see what the lead compensator contributes to the system by examining the Bode plot of $C_l(s)$:

We see that the phase plot of $C_l(s)$ has a hump, and we can use this positive phase contribution to increase the phase of $KP(s)$. Specifically, we would like to choose $\alpha$ and $p$ so that the hump occurs near the crossover frequency of $KC_l(s)P(s)$, thereby increasing the phase margin of the system. To see how to choose the pole and zero, note that the phase of $C_l(j\omega)$ is given by

$$\angle C_l(j\omega) = \tan^{-1}\left(\frac{j\omega}{\alpha p}\right) - \tan^{-1}\left(\frac{j\omega}{p}\right).$$

From the phase plot, we note that the maximum phase occurs halfway between the zero and the pole (on a logarithmic scale). If we denote the frequency where the maximum phase occurs as $\omega_{\text{max}}$, we have

$$\log \omega_{\text{max}} = \frac{1}{2} \left( \log(\alpha p) + \log(p) \right) = \log \sqrt{\alpha p^2},$$

from which we obtain $\omega_{\text{max}} = \sqrt{\alpha p}$. If we denote $\phi_{\text{max}} = \angle C_l(j\omega_{\text{max}})$, we obtain (after some algebra)

$$\sin \phi_{\text{max}} = \frac{1 - \alpha}{1 + \alpha}.$$

These expressions are important, so let’s restate them:
The maximum phase of the lead compensator with zero at \( \alpha p \) and pole at \( p \) is denoted by \( \phi_{\text{max}} \) and occurs at the frequency \( \omega_{\text{max}} = \sqrt{\alpha p} \). The maximum phase satisfies the equation

\[
\sin \phi_{\text{max}} = \frac{1 - \alpha}{1 + \alpha} \quad \text{or equivalently,} \quad \alpha = \frac{1 - \sin \phi_{\text{max}}}{1 + \sin \phi_{\text{max}}}
\]

The idea will be to choose the pole and zero of the compensator such that \( \omega_{\text{max}} \) lies on the crossover frequency of \( KP(s) \), with the hope of contributing an extra \( \phi_{\text{max}} \) degrees of phase margin. Let’s try an example to see how this works.

**Example.** Consider \( KP(s) = \frac{1}{s(s+1)} \). Draw an approximate Bode plot for \( C_l(s)KP(s) \) when we place the pole and zero of the compensator in such a way that the maximum compensator phase occurs at the gain crossover frequency of \( KP(s) \).

**Solution.**

From the above example, we see that although the compensator does contribute \( \phi_{\text{max}} \) to the phase at the gain crossover frequency of \( KP(s) \), the gain crossover frequency of \( C_l(s)KP(s) \) actually shifts to the right due to the positive magnitude contribution of \( C_l(s) \). Thus, the phase margin of \( KC_l(s)P(s) \) is actually a little less than the phase margin of \( KP(s) \) plus \( \phi_{\text{max}} \). In order to still get our desired phase margin, we should therefore make \( \phi_{\text{max}} \) a little larger than we need (usually about 10° extra is enough), so that the phase margin of \( KC_l(s)P(s) \) will meet the specification.

The complete design procedure for lead compensators is as follows.
1. Choose $K$ to meet a steady state error specification, or to meet a gain crossover frequency specification (in the latter case, one can choose $K$ so that the crossover frequency of $KP(s)$ is a little less than desired, since the lead compensator will shift the frequency to the right a little bit).

2. Find the phase margin of $KP(s)$.

3. Find how much extra phase is required in order to meet the phase margin spec. Set $\phi_{\text{max}}$ to be this extra phase plus $10^\circ$.

4. Find $\alpha$ from $\alpha = \frac{1 - \sin \phi_{\text{max}}}{1 + \sin \phi_{\text{max}}}$.

5. Set $\omega_{\text{max}}$ to be the gain crossover frequency of $KP(s)$. From this, we can calculate $p = \frac{\omega_{\text{max}}}{\sqrt{\alpha}}$ and $z = \sqrt{\alpha} \omega_{\text{max}}$.

6. Check if the compensator achieves the specifications. If not, iterate or add another lead compensator.

**Example.** Consider $P(s) = \frac{1}{s(s+1)}$. Design a lead compensator so that the closed loop system has a steady state tracking error of 0.1 to a ramp input, and overshoot less than 25%.

**Solution.**